Multivariable and Vector Calculus

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Preface

These notes started during the Spring of 2003. They are meant to be a gentle introduction to multivariable and vector calculus.

Throughout these notes I use Maple™ version 10 commands in order to illustrate some points of the theory.

I would appreciate any comments, suggestions, corrections, etc., which can be addressed to the email below.

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1 Vectors and Parametric Curves

1.1 Points and Vectors on the Plane

We start with a naïve introduction to some linear algebra necessary for the course. Those interested in more formal treatments can profit by reading [BlRo] or [Lan].

1 Definition (Scalar, Point, Bi-point, Vector) A *scalar* $\alpha \in \mathbb{R}$ is simply a real number. A *point* $\mathbf{r} \in \mathbb{R}^2$ is an ordered pair of real numbers, $\mathbf{r} = (x, y)$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Here the first coordinate x stipulates the location on the horizontal axis and the second coordinate y stipulates the location on the vertical axis. See figure 1.1. We will always denote the origin, that is, the point $(0, 0)$ by $\mathbf{O} = (0, 0)$. Given two points $\mathbf{r}$ and $\mathbf{r}'$ in $\mathbb{R}^2$ the directed line segment with departure point $\mathbf{r}$ and arrival point $\mathbf{r}'$ is called the *bi-point* $\mathbf{r}$, $\mathbf{r}'$[1] and is denoted by $[\mathbf{r}, \mathbf{r}']$. See figure 1.2 for an example. The bi-point $[\mathbf{r}, \mathbf{r}']$ can be thus interpreted as an arrow starting at $\mathbf{r}$ and finishing, with the arrow tip, at $\mathbf{r}'$. We say that $\mathbf{r}$ is the *tail* of the bi-point $[\mathbf{r}, \mathbf{r}']$ and that $\mathbf{r}'$ is its *head*. A *vector* $\overrightarrow{\mathbf{a}} \in \mathbb{R}^2$ is a codification of movement of a bi-point: given the bi-point $[\mathbf{r}, \mathbf{r}']$, we associate to it the vector $\overrightarrow{\mathbf{rr}'} = \begin{bmatrix} x' - x \\ y' - y \end{bmatrix}$ stipulating a movement of $x' - x$ units from (x, y) in the horizontal axis and of $y' - y$ units from the current position in the vertical axis. The *zero vector* $\overrightarrow{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ indicates no movement in either direction.

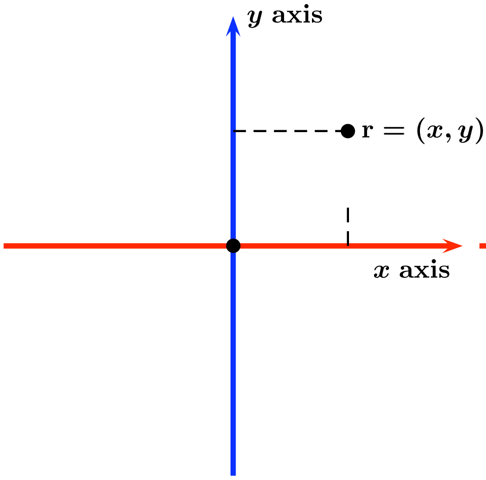

Figure 1.1: A point in $\mathbb{R}^2$.

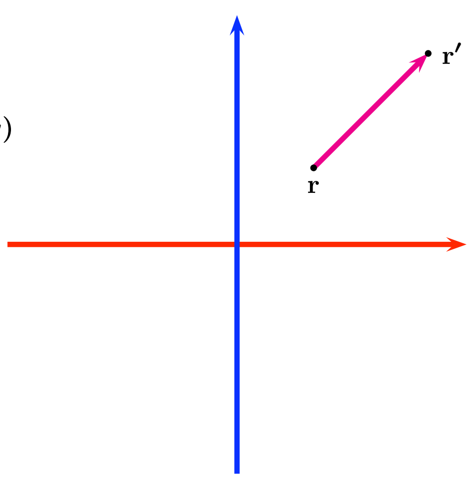

Figure 1.2: A bi-point in $\mathbb{R}^2$.

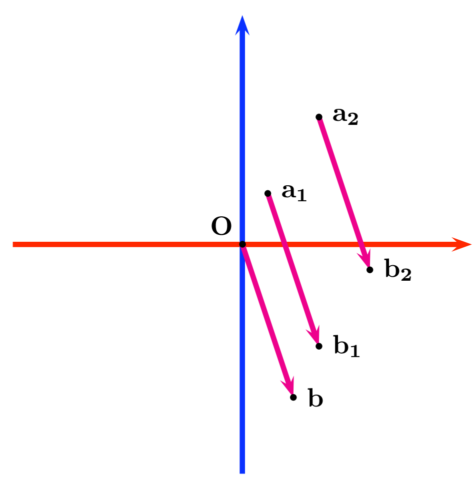

Figure 1.3: Example 2.

Notice that infinitely many different choices of departure and arrival points may give the same vector.

2 Example Consider the points

$$\mathbf{a}_1 = (1, 2), \quad \mathbf{b}_1 = (3, -4), \quad \mathbf{a}_2 = (3, 5), \quad \mathbf{b}_2 = (5, -1), \quad \mathbf{O} = (0, 0) \quad \mathbf{b} = (2, -6).$$

[1] Some authors use the terminology "*fixed vector*" instead of "bi-point."

Though the bi-points $[\mathbf{a_1}, \mathbf{b_1}]$, $[\mathbf{a_2}, \mathbf{b_2}]$ and $[\mathbf{O}, \mathbf{b}]$ are in different locations on the plane, they represent the same vector, as

$$\begin{bmatrix} 3-1 \\ -4-2 \end{bmatrix} = \begin{bmatrix} 5-3 \\ -1-5 \end{bmatrix} = \begin{bmatrix} 2-0 \\ -6-0 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

The instructions given by the vector are all the same: start at the point, go two units right and six units down. See figure 1.3.

In more technical language, a vector is an *equivalence class* of bi-points, that is, all bi-points that have the same length, have the same direction, and point in the same sense are equivalent, and the name of this equivalence is a *vector*. As an simple example of an equivalence class, consider the set of integers $\mathbb{Z}$. According to their remainder upon division by 3, each integer belongs to one of the three sets

$$3\mathbb{Z} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}, \quad 3\mathbb{Z}+1 = \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \quad 3\mathbb{Z}+2 = \{\ldots, -4, -1, 2, 5, 8, \ldots\}.$$

The equivalence class $3\mathbb{Z}$ comprises the integers divisible by 3, and for example, $-18 \in 3\mathbb{Z}$. Analogously, in example 2, the bi-point $[\mathbf{a_1}, \mathbf{b_1}]$ belongs to the equivalence class $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$, that is, $[\mathbf{a_1}, \mathbf{b_1}] \in \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

3 Definition The vector $\overrightarrow{\mathbf{Oa}}$ that corresponds to the point $\mathbf{a} \in \mathbb{R}^2$ is called the *position vector* of the point $\mathbf{a}$.

4 Definition Let $\mathbf{a} \neq \mathbf{b}$ be points on the plane and let $\overleftrightarrow{\mathbf{ab}}$ be the line passing through $\mathbf{a}$ and $\mathbf{b}$. The *direction* of the bi-point $[\mathbf{a}, \mathbf{b}]$ is the direction of the line L, that is, the angle $\theta \in [0; \pi[$ that the line $\overleftrightarrow{\mathbf{ab}}$ makes with the positive x-axis (horizontal axis), when measured counterclockwise. The direction of a vector $\overrightarrow{\mathbf{v}} \neq \overrightarrow{\mathbf{0}}$ is the direction of any of its bi-point representatives. See figure 1.4.

5 Definition We say that $[\mathbf{a}, \mathbf{b}]$ has the same direction as $[\mathbf{z}, \mathbf{w}]$ if $\overleftrightarrow{\mathbf{ab}} = \overleftrightarrow{\mathbf{zw}}$. We say that the bi-points $[\mathbf{a}, \mathbf{b}]$ and $[\mathbf{z}, \mathbf{w}]$ have the *same sense* if they have the same direction and if when translating one so as to its tail is over the other's tail, both their heads lie on the same half-plane made by the line perpendicular to then at their tails. They have *opposite sense* if they have the same direction and if when translating one so as to its tail is over the other's tail, their heads lie on different half-planes made by the line perpendicular to them at their tails. . See figures 1.5 and 1.6 . The *sense* of a vector is the sense of any of its bi-point representatives. Two bi-points are *parallel* if the lines containing them are parallel. Two vectors are parallel, if bi-point representatives of them are parallel.

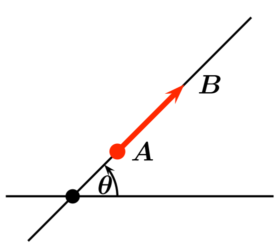

Figure 1.4: Direction of a bi-point

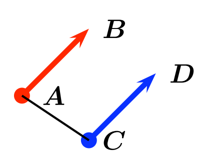

Figure 1.5: Bi-points with the same sense.

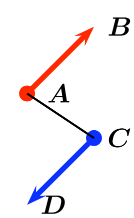

Figure 1.6: Bi-points with opposite sense.

☞ *Bi-point* $[\mathbf{b}, \mathbf{a}]$ *has the opposite sense of* $[\mathbf{a}, \mathbf{b}]$ *and so we write*

$$[\mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{b}].$$

Similarly we write, $\overrightarrow{\mathbf{ab}} = -\overrightarrow{\mathbf{ba}}$.

6 Definition The *Euclidean length or norm* of bi-point $[\mathbf{a}, \mathbf{b}]$ is simply the distance between $\mathbf{a}$ and $\mathbf{b}$ and it is denoted by

$$||[\mathbf{a}, \mathbf{b}]|| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

A bi-point is said to have *unit length* if it has norm 1. The *norm of a vector* is the norm of any of its bi-point representatives.

☞ *A vector is completely determined by three things: (i) its norm, (ii) its direction, and (iii) its sense. It is clear that the norm of a vector satisfies the following properties:*

1. $\left|\left|\overrightarrow{a}\right|\right| \geq 0$.
2. $\left|\left|\overrightarrow{a}\right|\right| = 0 \iff \overrightarrow{a} = \overrightarrow{0}$.

7 Example The vector $\overrightarrow{v} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ has norm $\left|\left|\overrightarrow{v}\right|\right| = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$.

8 Definition If $\overrightarrow{u}$ and $\overrightarrow{v}$ are two vectors in $\mathbb{R}^2$ their *vector sum* $\overrightarrow{u} + \overrightarrow{v}$ is defined by the coordinatewise addition

$$\overrightarrow{u} + \overrightarrow{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}. \tag{1.1}$$

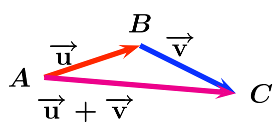

Figure 1.7: Addition of Vectors.

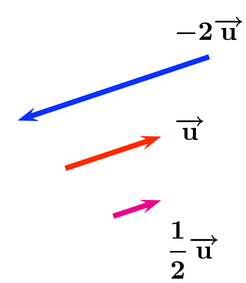

Figure 1.8: Scalar multiplication of vectors.

It is easy to see that vector addition is commutative and associative, that the vector $\overrightarrow{0}$ acts as an additive identity, and that the additive inverse of $\overrightarrow{a}$ is $-\overrightarrow{a}$. To add two vectors geometrically, proceed as follows. Draw a bi-point representative of $\overrightarrow{u}$. Find a bi-point representative of $\overrightarrow{v}$ having its tail at the tip of $\overrightarrow{u}$. The sum $\overrightarrow{u} + \overrightarrow{v}$ is the vector whose tail is that of the bi-point for $\overrightarrow{u}$ and whose tip is that of the bi-point for $\overrightarrow{v}$. In particular, if $\overrightarrow{u} = \overrightarrow{AB}$ and $\overrightarrow{v} = \overrightarrow{BC}$, then we have *Chasles' Rule*:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}. \tag{1.2}$$

See figures 1.7, 1.9, 1.10, and 1.11.

9 Definition If $\alpha \in \mathbb{R}$ and $\overrightarrow{a} \in \mathbb{R}^2$ we define *scalar multiplication* of a vector and a scalar by the coordinatewise multiplication

$$\alpha\overrightarrow{a} = \alpha \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix}. \tag{1.3}$$

It is easy to see that vector addition and scalar multiplication satisfies the following properties.

❶ $\alpha(\overrightarrow{a} + \overrightarrow{b}) = \alpha\overrightarrow{a} + \alpha\overrightarrow{b}$

❷ $(\alpha + \beta)\overrightarrow{a} = \alpha\overrightarrow{a} + \beta\overrightarrow{a}$

❸ $1\overrightarrow{a} = \overrightarrow{a}$

❹ $(\alpha\beta)\overrightarrow{a} = \alpha(\beta\overrightarrow{a})$

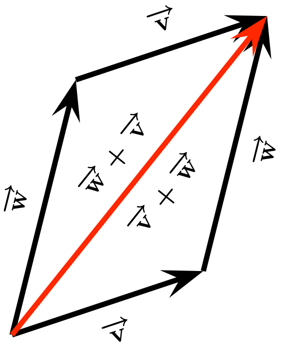

Figure 1.9: Commutativity

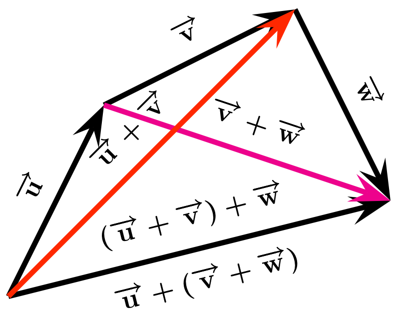

Figure 1.10: Associativity

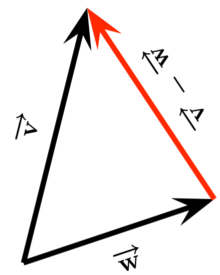

Figure 1.11: Difference

10 Definition Let $\overrightarrow{u} \neq \overrightarrow{0}$. Put $\mathbb{R}\overrightarrow{u} = \{\lambda\overrightarrow{u} : \lambda \in \mathbb{R}\}$ and let $\mathbf{a} \in \mathbb{R}^2$, The *affine line with direction vector* $\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *and passing through* $\mathbf{a}$ is the set of points on the plane

$$\mathbf{a} + \mathbb{R}\overrightarrow{u} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x = a_1 + tu_1, \quad y = a_2 + tu_2, \quad t \in \mathbb{R} \right\}.$$

See figure 1.12.

If $u_1 = 0$, the affine line defined above is vertical, as x is constant. If $u_1 \neq 0$, then

$$\frac{x - a_1}{u_1} = t \implies y = a_2 + \frac{(x - a_1)}{u_1}u_2 = \frac{u_2}{u_1}x + a_2 - a_1\frac{u_2}{u_1},$$

that is, the affine line is the Cartesian line with slope $\frac{u_2}{u_1}$. Conversely, if $y = mx + k$ is the equation of a Cartesian line, then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix} t + \begin{pmatrix} 0 \\ k \end{pmatrix},$$

that is, every Cartesian line is also an affine line and one may take the vector $\begin{bmatrix} 1 \\ m \end{bmatrix}$ as its direction vector. It also follows that two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are parallel if and only if the affine lines $\mathbb{R}\overrightarrow{u}$ and $\mathbb{R}\overrightarrow{v}$ are parallel. Hence, $\overrightarrow{u} \parallel \overrightarrow{v}$ if there exists a scalar $\lambda \in \mathbb{R}$ such that $\overrightarrow{u} = \lambda\overrightarrow{v}$.

☞ *Because* $\overrightarrow{0} = 0\overrightarrow{v}$ *for any vector* $\overrightarrow{v}$*, the* $\overrightarrow{0}$ *is parallel to every vector.*

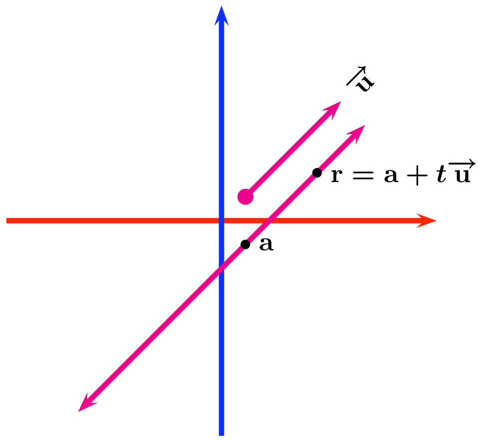

Figure 1.12: Parametric equation of a line on the plane.

11 Example Find a vector of length 3, parallel to $\overrightarrow{\mathbf{v}} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ but in the opposite sense.

Solution: ► *Since* $\left|\left|\overrightarrow{\mathbf{v}}\right|\right| = \sqrt{3}$, *the vector* $\dfrac{\overrightarrow{\mathbf{v}}}{\left|\left|\overrightarrow{\mathbf{v}}\right|\right|}$ *has unit norm and has the same direction and sense as* $\overrightarrow{\mathbf{v}}$, *and so the vector sought is*

$$-3\frac{\overrightarrow{\mathbf{v}}}{\left|\left|\overrightarrow{\mathbf{v}}\right|\right|} = -\frac{3}{\sqrt{3}}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ -\sqrt{6} \end{bmatrix}.$$

◄

12 Example Find the parametric equation of the line passing through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and in the direction of the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Solution: ► *The desired equation is plainly*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t\begin{bmatrix} 2 \\ 3 \end{bmatrix} \implies x = 1 + 2t, \quad y = -1 + 3t, \quad t \in \mathbb{R}.$$

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Some plane geometry results can be easily proved by means of vectors. Here are some examples.

13 Example Given a pentagon $ABCDE$, determine the vector sum $\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}} + \overrightarrow{\mathbf{CD}} + \overrightarrow{\mathbf{DE}} + \overrightarrow{\mathbf{EA}}$.

Solution: ► *Utilising Chasles' Rule several times:*

$$\overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{AA}} = \overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}} + \overrightarrow{\mathbf{CD}} + \overrightarrow{\mathbf{DE}} + \overrightarrow{\mathbf{EA}}.$$

. ◄

14 Example Consider a $\triangle \mathbf{ABC}$. Demonstrate that the line segment joining the midpoints of two sides is parallel to the third side and it is in fact, half its length.

Solution: ► *Let the midpoints of* $[\mathbf{A}, \mathbf{B}]$ *and* $[\mathbf{C}, \mathbf{A}]$, *be* $\mathbf{M_C}$ *and* $\mathbf{M_B}$, *respectively. We will demonstrate that* $\overrightarrow{\mathbf{BC}} = 2\overrightarrow{\mathbf{M_C M_B}}$. *We have,* $2\overrightarrow{\mathbf{AM_C}} = \overrightarrow{\mathbf{AB}}$ *and* $2\overrightarrow{\mathbf{AM_B}} = \overrightarrow{\mathbf{AC}}$. *Therefore,*

$$\begin{aligned}
\overrightarrow{\mathbf{BC}} &= \overrightarrow{\mathbf{BA}} + \overrightarrow{\mathbf{AC}} \\
&= -\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{AC}} \\
&= -2\overrightarrow{\mathbf{AM_C}} + 2\overrightarrow{\mathbf{AM_B}} \\
&= 2\overrightarrow{\mathbf{M_C A}} + 2\overrightarrow{\mathbf{AM_B}} \\
&= 2(\overrightarrow{\mathbf{M_C A}} + \overrightarrow{\mathbf{AM_B}}) \\
&= 2\overrightarrow{\mathbf{M_C M_B}},
\end{aligned}$$

as we were to shew. ◄

15 Example In $\triangle ABC$, let $\mathbf{M_C}$ be the midpoint of $[\mathbf{A}, \mathbf{B}]$. Demonstrate that

$$\overrightarrow{\mathbf{CM_C}} = \frac{1}{2}\left(\overrightarrow{\mathbf{CA}} + \overrightarrow{\mathbf{CB}}\right).$$

Solution: ► *As* $\overrightarrow{\mathbf{AM_C}} = \overrightarrow{\mathbf{M_C B}}$, *we have,*

$$\begin{aligned}
\overrightarrow{\mathbf{CA}} + \overrightarrow{\mathbf{CB}} &= \overrightarrow{\mathbf{CM_C}} + \overrightarrow{\mathbf{M_C A}} + \overrightarrow{\mathbf{CM_C}} + \overrightarrow{\mathbf{M_C B}} \\
&= 2\overrightarrow{\mathbf{CM_C}} - \overrightarrow{\mathbf{AM_C}} + \overrightarrow{\mathbf{M_C B}} \\
&= 2\overrightarrow{\mathbf{CM_C}},
\end{aligned}$$

from where the result follows. ◄

16 Example If the medians $[\mathbf{A}, \mathbf{M_A}]$ and $[\mathbf{B}, \mathbf{M_B}]$ of the non-degenerate $\triangle \mathbf{ABC}$ intersect at the point $\mathbf{G}$, demonstrate that

$$\overrightarrow{\mathbf{AG}} = 2\overrightarrow{\mathbf{GM_A}}; \qquad \overrightarrow{\mathbf{BG}} = 2\overrightarrow{\mathbf{GM_B}}.$$

See figure 1.13.

Solution: ► *Since the triangle is non-degenerate, the lines* $\overleftrightarrow{\mathbf{AM_A}}$ *and* $\overleftrightarrow{\mathbf{BM_B}}$ *are not parallel, and hence meet at a point* $\mathbf{G}$. *Therefore,* $\overrightarrow{\mathbf{AG}}$ *and* $\overrightarrow{\mathbf{GM_A}}$ *are parallel and hence there is a scalar* a *such that* $\overrightarrow{\mathbf{AG}} = a\overrightarrow{\mathbf{GM_A}}$. *In the same fashion, there is a scalar* b *such that* $\overrightarrow{\mathbf{BG}} = b\overrightarrow{\mathbf{GM_B}}$. *From example 14,*

$$\begin{aligned}
2\overrightarrow{\mathbf{M_A M_B}} &= \overrightarrow{\mathbf{BA}} \\
&= \overrightarrow{\mathbf{BG}} + \overrightarrow{\mathbf{GA}} \\
&= b\overrightarrow{\mathbf{GM_B}} - a\overrightarrow{\mathbf{GM_A}} \\
&= b\overrightarrow{\mathbf{GM_A}} + b\overrightarrow{\mathbf{M_A M_B}} - a\overrightarrow{\mathbf{GM_A}},
\end{aligned}$$

and thus

$$(2-b)\overrightarrow{\mathbf{M_A M_B}} = (b-a)\overrightarrow{\mathbf{GM_A}}.$$

Since $\triangle \mathbf{ABC}$ *is non-degenerate,* $\overrightarrow{\mathbf{M_A M_B}}$ *and* $\overrightarrow{\mathbf{GM_A}}$ *are not parallel, whence*

$$2 - b = 0, \quad b - a = 0, \quad \Longrightarrow a = b = 2.$$

◄

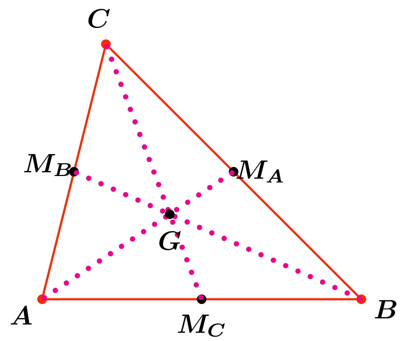

Figure 1.13: Example 17.

17 Example The medians of a non-degenerate triangle $\triangle ABC$ are concurrent. The point of concurrency $\mathbf{G}$ is called the *barycentre* or *centroid* of the triangle. See figure 1.13.

Solution: ► *Let* $\mathbf{G}$ *be as in example 16. We must shew that the line* $\overleftrightarrow{\mathbf{CM_C}}$ *also passes through* $\mathbf{G}$. *Let the line* $\overleftrightarrow{\mathbf{CM_C}}$ *and* $\overleftrightarrow{\mathbf{BM_B}}$ *meet in* $\mathbf{G}'$. *By the aforementioned example,*

$$\overrightarrow{\mathbf{AG}} = 2\overrightarrow{\mathbf{GM_A}}; \qquad \overrightarrow{\mathbf{BG}} = 2\overrightarrow{\mathbf{GM_B}}; \quad \overrightarrow{\mathbf{BG'}} = 2\overrightarrow{\mathbf{G'M_B}}; \qquad \overrightarrow{\mathbf{CG'}} = 2\overrightarrow{\mathbf{G'M_C}}.$$

It follows that

$$\begin{aligned}\overrightarrow{\mathbf{GG'}} &= \overrightarrow{\mathbf{GB}} + \overrightarrow{\mathbf{BG'}} \\ &= -2\overrightarrow{\mathbf{GM_B}} + 2\overrightarrow{\mathbf{G'M_B}} \\ &= 2(\overrightarrow{\mathbf{M_BG}} + \overrightarrow{\mathbf{G'M_B}}) \\ &= 2\overrightarrow{\mathbf{G'G}}.\end{aligned}$$

Therefore

$$\overrightarrow{\mathbf{GG'}} = -2\overrightarrow{\mathbf{GG'}} \implies 3\overrightarrow{\mathbf{GG'}} = \overrightarrow{0} \implies \overrightarrow{\mathbf{GG'}} = \overrightarrow{0} \implies G = G',$$

demonstrating the result. ◄

Homework

Problem 1.1.1 Is there is any truth to the statement "a vector is that which has magnitude and direction"?

Problem 1.1.2 $ABCD$ is a parallelogram. E is the midpoint of $[\mathbf{B}, \mathbf{C}]$ and F is the midpoint of $[\mathbf{D}, \mathbf{C}]$. Prove that

$$\overrightarrow{\mathbf{AC}} + \overrightarrow{\mathbf{BD}} = 2\overrightarrow{\mathbf{BC}}.$$

Problem 1.1.3 (Varignon's Theorem) Use vector algebra in order to prove that in any quadrilateral $\mathbf{ABCD}$, whose sides do not intersect, the quadrilateral formed by the midpoints of the sides is a parallelogram.

Problem 1.1.4 Let $\mathbf{A}, \mathbf{B}$ be two points on the plane. Construct two points $\mathbf{I}$ and $\mathbf{J}$ such that

$$\overrightarrow{\mathbf{IA}} = -3\overrightarrow{\mathbf{IB}}, \quad \overrightarrow{\mathbf{JA}} = -\frac{1}{3}\overrightarrow{\mathbf{JB}},$$

and then demonstrate that for any arbitrary point $\mathbf{M}$ on the plane

$$\overrightarrow{\mathbf{MA}} + 3\overrightarrow{\mathbf{MB}} = 4\overrightarrow{\mathbf{MI}}$$

and

$$3\overrightarrow{\mathbf{MA}} + \overrightarrow{\mathbf{MB}} = 4\overrightarrow{\mathbf{MJ}}.$$

Problem 1.1.5 Find the Cartesian equation corresponding to the line with parametric equation

$$x = -1 + t, \qquad y = 2 - t.$$

Problem 1.1.6 Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be points on the plane with $\mathbf{x} \neq \mathbf{y}$ and consider $\triangle\mathbf{xyz}$. Let $\mathbf{Q}$ be a point on side $[\mathbf{x}, \mathbf{z}]$ such that $||[\mathbf{x}, \mathbf{Q}]|| : ||[\mathbf{Q}, \mathbf{z}]|| = 3 : 4$ and let $\mathbf{P}$ be a point on $[\mathbf{y}, \mathbf{z}]$ such that $||[\mathbf{y}, \mathbf{P}]|| : ||[\mathbf{P}, \mathbf{Q}]|| = 7 : 2$. Let $\mathbf{T}$ be an arbitrary point on the plane.

1. Find rational numbers α and β such that $\overrightarrow{\mathbf{TQ}} = \alpha\overrightarrow{\mathbf{Tx}} + \beta\overrightarrow{\mathbf{Tz}}$.

2. Find rational numbers l, m, n such that $\overrightarrow{\mathbf{TP}} = l\overrightarrow{\mathbf{Tx}} + m\overrightarrow{\mathbf{Ty}} + n\overrightarrow{\mathbf{Tz}}$.

Problem 1.1.7 Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be points on the plane with $\mathbf{x} \neq \mathbf{y}$. Demonstrate that

1. The point $\mathbf{a}$ belongs to the line $\overleftrightarrow{\mathbf{xy}}$ if and only if there exists scalars α, β with $\alpha + \beta = 1$ such that
$$\overrightarrow{\mathbf{za}} = \alpha\overrightarrow{\mathbf{zx}} + \beta\overrightarrow{\mathbf{zy}}.$$
2. The point $\mathbf{a}$ belongs to the line segment $[\mathbf{x}; \mathbf{y}]$ if and only if there exists scalars $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta = 1$ such that
$$\overrightarrow{\mathbf{za}} = \alpha\overrightarrow{\mathbf{zx}} + \beta\overrightarrow{\mathbf{zy}}.$$
3. The point $\mathbf{a}$ belongs to the interior of the triangle $\triangle\mathbf{xyz}$ if and only if there exists scalars $\alpha > 0, \beta > 0$ with $\alpha + \beta < 1$ such that
$$\overrightarrow{\mathbf{za}} = \alpha\overrightarrow{\mathbf{zx}} + \beta\overrightarrow{\mathbf{zy}}.$$

Problem 1.1.8 A circle is divided into three, four equal, or six equal parts (figures 1.17 through 1.19). Find the sum of the vectors. Assume that the divisions start or stop at the centre of the circle, as suggested in the figures.

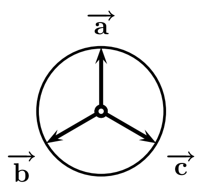

Figure 1.14: [A]. Problem 1.1.8.

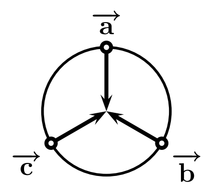

Figure 1.15: [B]. Problem 1.1.8.

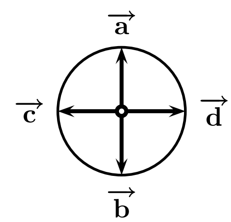

Figure 1.16: [C]. Problem 1.1.8.

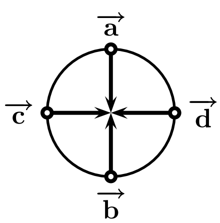

Figure 1.17: [D]. Problem 1.1.8.

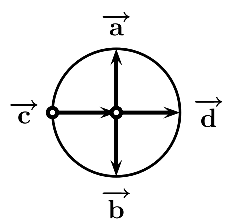

Figure 1.18: [E]. Problem 1.1.8.

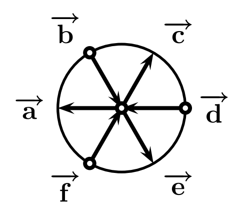

Figure 1.19: [F]. Problem 1.1.8.

1.2 Scalar Product on the Plane

We will now define an operation between two plane vectors that will provide a further tool to examine the geometry on the plane.

18 Definition Let $\overrightarrow{x} \in \mathbb{R}^2$ and $\overrightarrow{y} \in \mathbb{R}^2$. Their *scalar product* (*dot product*, *inner product*) is defined and denoted by
$$\overrightarrow{x} \bullet \overrightarrow{y} = x_1y_1 + x_2y_2.$$

19 Example If $\overrightarrow{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, then $\overrightarrow{a} \bullet \overrightarrow{b} = 1 \cdot 3 + 2 \cdot 4 = 11$.

The following properties of the scalar product are easy to deduce from the definition.

SP1 **Bilinearity**

$$(\overrightarrow{x} + \overrightarrow{y})\bullet\overrightarrow{z} = \overrightarrow{x}\bullet\overrightarrow{z} + \overrightarrow{y}\bullet\overrightarrow{z}, \quad \overrightarrow{x}\bullet(\overrightarrow{y} + \overrightarrow{z}) = \overrightarrow{x}\bullet\overrightarrow{y} + \overrightarrow{x}\bullet\overrightarrow{z} \tag{1.4}$$

SP2 **Scalar Homogeneity**

$$(\alpha\overrightarrow{x})\bullet\overrightarrow{y} = \overrightarrow{x}\bullet(\alpha\overrightarrow{y}) = \alpha(\overrightarrow{x}\bullet\overrightarrow{y}),\ \alpha \in \mathbb{R}. \tag{1.5}$$

SP3 **Commutativity**

$$\overrightarrow{x}\bullet\overrightarrow{y} = \overrightarrow{y}\bullet\overrightarrow{x} \tag{1.6}$$

SP4

$$\overrightarrow{x}\bullet\overrightarrow{x} \ \geq\ 0 \tag{1.7}$$

SP5

$$\overrightarrow{x}\bullet\overrightarrow{x} = 0 \iff \overrightarrow{x} = \overrightarrow{0} \tag{1.8}$$

SP6

$$||\overrightarrow{x}|| = \sqrt{\overrightarrow{x}\bullet\overrightarrow{x}} \tag{1.9}$$

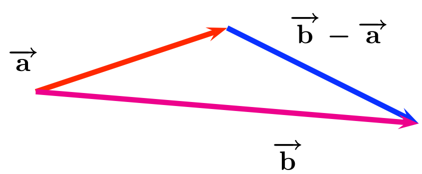

Figure 1.20: Theorem 21.

20 Definition Given vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, we define the (convex) angle between them, denoted by $(\widehat{\overrightarrow{a}, \overrightarrow{b}}) \in [0; \pi]$, as the angle between the affine lines $\mathbb{R}\overrightarrow{a}$ and $\mathbb{R}\overrightarrow{b}$.

21 Theorem Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be vectors in $\mathbb{R}^2$. Then

$$\overrightarrow{a}\bullet\overrightarrow{b} = ||\overrightarrow{a}||||\overrightarrow{b}||\cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}).$$

Proof: *From figure 1.20, using Al-Kashi's Law of Cosines on the length of the vectors, and (1.4) through (1.9) we have*

$$\begin{aligned}
&||\overrightarrow{b} - \overrightarrow{a}||^2 = ||\overrightarrow{a}||^2 + ||\overrightarrow{b}||^2 - 2||\overrightarrow{a}||||\overrightarrow{b}||\cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}) \\
\iff\ & (\overrightarrow{b} - \overrightarrow{a})\bullet(\overrightarrow{b} - \overrightarrow{a}) = ||\overrightarrow{a}||^2 + ||\overrightarrow{b}||^2 - 2||\overrightarrow{a}||||\overrightarrow{b}||\cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}) \\
\iff\ & \overrightarrow{b}\bullet\overrightarrow{b} - 2\overrightarrow{a}\bullet\overrightarrow{b} + \overrightarrow{a}\bullet\overrightarrow{a} = ||\overrightarrow{a}||^2 + ||\overrightarrow{b}||^2 - 2||\overrightarrow{a}||||\overrightarrow{b}||\cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}) \\
\iff\ & ||\overrightarrow{b}||^2 - 2\overrightarrow{a}\bullet\overrightarrow{b} + ||\overrightarrow{a}||^2 = ||\overrightarrow{a}||^2 + ||\overrightarrow{b}||^2 - 2||\overrightarrow{a}||||\overrightarrow{b}||\cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}) \\
\iff\ & \overrightarrow{a}\bullet\overrightarrow{b} = ||\overrightarrow{a}||||\overrightarrow{b}||\cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}),
\end{aligned}$$

as we wanted to shew. ❑

Putting $(\widehat{\overrightarrow{a}, \overrightarrow{b}}) = \dfrac{\pi}{2}$ in Theorem 21 we obtain the following corollary.

22 Corollary Two vectors in $\mathbb{R}^2$ are perpendicular if and only if their dot product is 0.

☞ *It follows that the vector $\overrightarrow{0}$ is simultaneously parallel and perpendicular to any vector!*

23 Definition Two vectors are said to be *orthogonal* if they are perpendicular. If $\overrightarrow{a}$ is orthogonal to $\overrightarrow{b}$, we write $\overrightarrow{a} \perp \overrightarrow{b}$.

24 Definition If $\overrightarrow{a} \perp \overrightarrow{b}$ and $||\overrightarrow{a}|| = \left|\left|\overrightarrow{b}\right|\right| = 1$ we say that $\overrightarrow{a}$ and $\overrightarrow{b}$ are *orthonormal.*

Since $|\cos\theta| \leq 1$ we also have

25 Corollary (Cauchy-Bunyakovsky-Schwarz Inequality)

$$\left|\overrightarrow{a}\bullet\overrightarrow{b}\right| \leq ||\overrightarrow{a}||\left|\left|\overrightarrow{b}\right|\right|.$$

Equality occurs if and only if $\overrightarrow{a} \parallel \overrightarrow{b}$.

If $\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, the CBS Inequality takes the form

$$|a_1b_1 + a_2b_2| \leq (a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2}. \tag{1.10}$$

26 Example Let a, b be positive real numbers. Minimise $a^2 + b^2$ subject to the constraint $a + b = 1$.

Solution: ► *By the CBS Inequality,*

$$1 = |a \cdot 1 + b \cdot 1| \leq (a^2 + b^2)^{1/2}(1^2 + 1^2)^{1/2} \implies a^2 + b^2 \geq \frac{1}{2}.$$

Equality occurs if and only if $\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *In such case,* $a = b = \lambda$, *and so equality is achieved for* $a = b = \frac{1}{2}$. ◄

27 Corollary (Triangle Inequality)

$$\left|\left|\overrightarrow{a} + \overrightarrow{b}\right|\right| \leq ||\overrightarrow{a}|| + \left|\left|\overrightarrow{b}\right|\right|.$$

Proof:

$$\begin{aligned} ||\overrightarrow{a} + \overrightarrow{b}||^2 &= (\overrightarrow{a} + \overrightarrow{b})\bullet(\overrightarrow{a} + \overrightarrow{b}) \\ &= \overrightarrow{a}\bullet\overrightarrow{a} + 2\,\overrightarrow{a}\bullet\overrightarrow{b} + \overrightarrow{b}\bullet\overrightarrow{b} \\ &\leq ||\overrightarrow{a}||^2 + 2||\overrightarrow{a}||||\overrightarrow{b}|| + ||\overrightarrow{b}||^2 \\ &= (||\overrightarrow{a}|| + ||\overrightarrow{b}||)^2, \end{aligned}$$

from where the desired result follows. ❑

28 Example Let x, y, z be positive real numbers. Prove that

$$\sqrt{2}(x + y + z) \leq \sqrt{x^2 + y^2} + \sqrt{y^2 + z^2} + \sqrt{z^2 + x^2}.$$

Solution: ► *Put* $\overrightarrow{a} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\overrightarrow{b} = \begin{bmatrix} y \\ z \end{bmatrix}$, $\overrightarrow{c} = \begin{bmatrix} z \\ x \end{bmatrix}$. *Then*

$$\left|\left|\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}\right|\right| = \left|\left|\begin{bmatrix} x + y + z \\ x + y + z \end{bmatrix}\right|\right| = \sqrt{2}(x + y + z).$$

Also,

$$||\overrightarrow{\mathbf{a}}|| + \left|\left|\overrightarrow{\mathbf{b}}\right|\right| + ||\overrightarrow{\mathbf{c}}|| = \sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{z^2+x^2},$$

and the assertion follows by the triangle inequality

$$\left|\left|\overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{c}}\right|\right| \leq ||\overrightarrow{\mathbf{a}}|| + \left|\left|\overrightarrow{\mathbf{b}}\right|\right| + ||\overrightarrow{\mathbf{c}}||.$$

◀

We now use vectors to prove a classical theorem of Euclidean geometry.

29 Definition Let A and B be points on the plane and let $\overrightarrow{\mathbf{u}}$ be a unit vector. If $\overrightarrow{\mathbf{AB}} = \lambda\overrightarrow{\mathbf{u}}$, then λ is the *directed distance or algebraic measure of the line segment* $[AB]$ *with respect to the vector* $\overrightarrow{\mathbf{u}}$. We will denote this distance by $\overline{AB}_{\overrightarrow{\mathbf{u}}}$, or more routinely, if the vector $\overrightarrow{\mathbf{u}}$ is patent, by $\overline{AB}$. Observe that $\overline{AB} = -\overline{BA}$.

30 Theorem (Thales' Theorem) Let $\overleftrightarrow{\mathbf{D}}$ y $\overleftrightarrow{\mathbf{D}'}$ be two distinct lines on the plane. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be distinct points of $\overleftrightarrow{\mathbf{D}}$, and $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ be distinct points of $\overleftrightarrow{\mathbf{D}'}$, $\mathbf{A} \neq \mathbf{A}'$, $\mathbf{B} \neq \mathbf{B}'$, $\mathbf{C} \neq \mathbf{C}'$, $\mathbf{A} \neq \mathbf{B}$, $\mathbf{A}' \neq \mathbf{B}'$. Let $\overleftrightarrow{\mathbf{AA}'} \parallel \overleftrightarrow{\mathbf{BB}'}$. Then

$$\overleftrightarrow{\mathbf{AA}'} \parallel \overleftrightarrow{\mathbf{CC}'} \iff \frac{\overline{AC}}{\overline{AB}} = \frac{\overline{A'C'}}{\overline{A'B'}}.$$

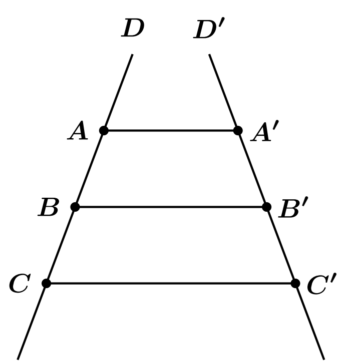

Figure 1.21: Thales' Theorem.

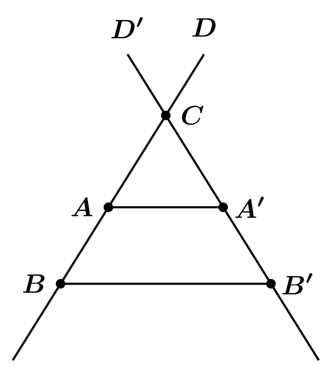

Figure 1.22: Corollary to Thales' Theorem.

Proof: *Refer to figure 1.2. On the one hand, because they are unit vectors in the same direction,*

$$\frac{\overrightarrow{\mathbf{AB}}}{\overline{AB}} = \frac{\overrightarrow{\mathbf{AC}}}{\overline{AC}}; \qquad \frac{\overrightarrow{\mathbf{A'B'}}}{\overline{A'B'}} = \frac{\overrightarrow{\mathbf{A'C'}}}{\overline{A'C'}}.$$

On the other hand, by Chasles' Rule,

$$\overrightarrow{\mathbf{BB}'} = \overrightarrow{\mathbf{BA}} + \overrightarrow{\mathbf{AA}'} + \overrightarrow{\mathbf{A'B'}} = (\overrightarrow{\mathbf{A'B'}} - \overrightarrow{\mathbf{AB}}) + \overrightarrow{\mathbf{AA}'}.$$

Since $\overleftrightarrow{\mathbf{AA}'} \parallel \overleftrightarrow{\mathbf{BB}'}$, *there is a scalar* $\lambda \in \mathbb{R}$ *such that*

$$\overrightarrow{\mathbf{A'B'}} = \overrightarrow{\mathbf{AB}} + \lambda\overrightarrow{\mathbf{AA}'}.$$

Assembling these results,

$$\begin{aligned} \overrightarrow{\mathbf{CC}'} &= \overrightarrow{\mathbf{CA}} + \overrightarrow{\mathbf{AA}'} + \overrightarrow{\mathbf{A'C'}} \\ &= -\frac{\overline{AC}}{\overline{AB}} \cdot \overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{AA}'} + \frac{\overline{A'C'}}{\overline{A'B'}}\left(\overrightarrow{\mathbf{AB}} + \lambda\overrightarrow{\mathbf{AA}'}\right) \\ &= \left(\frac{\overline{A'C'}}{\overline{A'B'}} - \frac{\overline{AC}}{\overline{AB}}\right)\overrightarrow{\mathbf{AB}} + \left(1 + \lambda\frac{\overline{A'C'}}{\overline{A'B'}}\right)\overrightarrow{\mathbf{AA}'}. \end{aligned}$$

As the line $\overleftrightarrow{AA'}$ is not parallel to the line $\overleftrightarrow{AB}$, the equality above reveals that

$$\overleftrightarrow{AA'} \parallel \overleftrightarrow{CC'} \iff \frac{\overline{AC}}{\overline{AB}} - \frac{\overline{A'C'}}{\overline{A'B'}} = 0,$$

proving the theorem.❑

From the preceding theorem, we immediately gather the following corollary. (See figure 1.2.)

31 Corollary Let $\overleftrightarrow{D}$ and $\overleftrightarrow{D'}$ are distinct lines, intersecting in the unique point C. Let A, B, be points on line $\overleftrightarrow{D}$, and A', B', points on line $\overleftrightarrow{D'}$. Then

$$\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'} \iff \frac{\overline{CB}}{\overline{CA}} = \frac{\overline{CB'}}{\overline{CA'}}.$$

1.3 Linear Independence

Consider now two arbitrary vectors in $\mathbb{R}^2$, $\overrightarrow{x}$ and $\overrightarrow{y}$, say. Under which conditions can we write an arbitrary vector $\overrightarrow{v}$ on the plane as a linear combination of $\overrightarrow{x}$ and $\overrightarrow{y}$, that is, when can we find scalars a, b such that

$$\overrightarrow{v} = a\overrightarrow{x} + b\overrightarrow{y}\ ?$$

The answer can be promptly obtained algebraically. Operating formally,

$$\begin{aligned} \overrightarrow{v} = a\overrightarrow{x} + b\overrightarrow{y} \quad &\iff \quad v_1 = ax_1 + by_1, \qquad v_2 = ax_2 + by_2 \\ &\iff \quad a = \frac{v_1y_2 - v_2y_1}{x_1y_2 - x_2y_1}, \qquad b = \frac{x_1v_2 - x_2v_1}{x_1y_2 - x_2y_1}. \end{aligned}$$

The above expressions for a and b make sense only if $x_1y_2 \neq x_2y_1$. But, what does it mean $x_1y_2 = x_2y_1$? If none of these are zero then $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \lambda$, say, and to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \iff \overrightarrow{x} \parallel \overrightarrow{y}.$$

If $x_1 = 0$, then either $x_2 = 0$ or $y_1 = 0$. In the first case, $\overrightarrow{x} = \overrightarrow{0}$, and *a fortiori* $\overrightarrow{x} \parallel \overrightarrow{y}$, since all vectors are parallel to the zero vector. In the second case we have

$$\overrightarrow{x} = x_2\overrightarrow{j}, \qquad \overrightarrow{y} = y_2\overrightarrow{j},$$

and so both vectors are parallel to $\overrightarrow{j}$ and hence $\overrightarrow{x} \parallel \overrightarrow{y}$. We have demonstrated the following theorem.

32 Theorem Given two vectors in $\mathbb{R}^2$, $\overrightarrow{x}$ and $\overrightarrow{y}$, an arbitrary vector $\overrightarrow{v}$ can be written as the linear combination

$$\overrightarrow{v} = a\overrightarrow{x} + b\overrightarrow{y}, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}$$

if and only if $\overrightarrow{x}$ is not parallel to $\overrightarrow{y}$. In this last case we say that $\overrightarrow{x}$ is *linearly independent* from vector $\overrightarrow{y}$. If two vectors are not linearly independent, then we say that they are *linearly dependent*.

33 Example The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are clearly linearly independent, since one is not a scalar multiple of the other. Given an arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix}$ we can express it as a linear combination of these vectors as follows:

$$\begin{bmatrix} a \\ b \end{bmatrix} = (a - b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Consider now two linearly independent vectors $\overrightarrow{x}$ and $\overrightarrow{y}$. For $a \in [0;1]$, $a\overrightarrow{x}$ is parallel to $\overrightarrow{x}$ and traverses the whole length of $\overrightarrow{x}$: from its tip (when $a = 1$) to its tail (when $a = 0$). In the same manner, for $b \in [0;1]$, $b\overrightarrow{y}$ is parallel to $\overrightarrow{y}$ and traverses the whole length of $\overrightarrow{y}$. The linear combination $a\overrightarrow{x} + b\overrightarrow{y}$ is also a vector on the plane.

34 Definition Given two linearly independent vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ consider bi-point representatives of them with the tails at the origin. The *fundamental parallelogram* of the the vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ is the set

$$\{a\overrightarrow{x} + b\overrightarrow{y} : a \in [0;1], b \in [0;1]\}.$$

Figure 1.23 shews the fundamental parallelogram of $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$, coloured in brown, and the respective tiling of the plane by various translations of it. Observe that the vertices of this parallelogram are $\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$. In essence then, linear independence of two vectors on the plane means that we may obtain every vector on the plane as a linear combination of these two vectors and hence cover the whole plane by all these linear combinations.

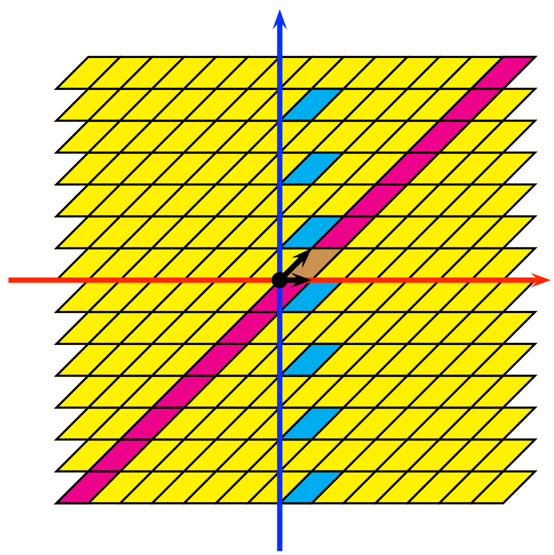

Figure 1.23: Tiling and the fundamental parallelogram.

Homework

Problem 1.3.1 Prove that $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ are linearly independent, and draw their fundamental parallelogram.

Problem 1.3.2 Write an arbitrary vector $\begin{bmatrix}a\\b\end{bmatrix}$ on the plane, as a linear combination of the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$.

Problem 1.3.3 Consider the line with Cartesian equation $L : ax + by = c$, where not both of a, b are zero. Let $\mathbf{t}$ be a point not on L. Find a formula for the distance from $\mathbf{t}$ to L.

Problem 1.3.4 Prove that two non-zero perpendicular vectors in $\mathbb{R}^2$ must be linearly independent.

1.4 Geometric Transformations in two dimensions

We now are interested in the following fundamental functions of sets (figures) on the plane: translations, scalings (stretching or shrinking) reflexions about the axes, and rotations about the origin. It will turn out that a handy tool for investigating all of these (with the exception of translations), will be certain construct called *matrices* which we will study in the next section.

First observe what is meant by a function $F : \mathbb{R}^2 \to \mathbb{R}^2$. This means that the input of the function is a point of the plane, and the output is also a point on the plane.
A rather uninteresting example, but nevertheless an important one is the following.

35 Example The function $\mathbf{I} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{I}(\mathbf{x}) = \mathbf{x}$ is called the *identity transformation*. Observe that the identity transformation leaves a point untouched.

We start with the simplest of these functions.

36 Definition A function $T_{\overrightarrow{v}} : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *translation* if it is of the form $T_{\overrightarrow{v}}(\mathbf{x}) = \mathbf{x} + \overrightarrow{v}$, where $\overrightarrow{v}$ is a fixed vector on the plane.

A translation simply shifts an object on the plane rigidly (that is, it does not distort it shape or re-orient it), to a copy of itself a given amount of units from where it was. See figure 1.24 for an example.

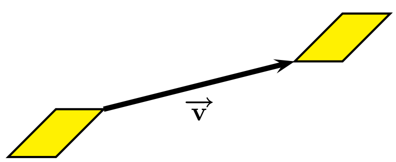

Figure 1.24: A translation.

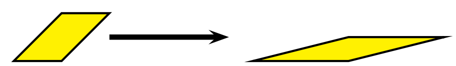

Figure 1.25: A scaling.

It is clear that the composition of any two translations commutes, that is, if $T_{\overrightarrow{v}_1}, T_{\overrightarrow{v}_2} : \mathbb{R}^2 \to \mathbb{R}^2$ are translations, then $T_{\overrightarrow{v}_1} \circ T_{\overrightarrow{v}_2} = T_{\overrightarrow{v}_2} \circ T_{\overrightarrow{v}_1}$. For let $T_1(\mathbf{a}) = \mathbf{a} + \overrightarrow{v}_1$ and $T_{\overrightarrow{v}_2}(\mathbf{a}) = \mathbf{a} + \overrightarrow{v}_2$. Then

$$(T_{\overrightarrow{v}_1} \circ T_{\overrightarrow{v}_2})(\mathbf{a}) = T_{\overrightarrow{v}_1}(T_{\overrightarrow{v}_2}(\mathbf{a})) = T_{\overrightarrow{v}_2}(\mathbf{a}) + \overrightarrow{v}_1 = \mathbf{a} + \overrightarrow{v}_2 + \overrightarrow{v}_1,$$

and

$$(T_{\overrightarrow{v}_2} \circ T_{\overrightarrow{v}_1})(\mathbf{a}) = T_{\overrightarrow{v}_2}(T_{\overrightarrow{v}_1}(\mathbf{a})) = T_{\overrightarrow{v}_1}(\mathbf{a}) + \overrightarrow{v}_2 = \mathbf{a} + \overrightarrow{v}_1 + \overrightarrow{v}_2,$$

from where the commutativity claim is deduced.

37 Definition A function $S_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *scaling* if it is of the form $S_{a,b}(\mathbf{r}) = \begin{pmatrix} ax \\ by \end{pmatrix}$, where $a > 0$, $b > 0$ are real numbers.

Figure 1.25 shews the scaling $S_{2,0.5}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ 0.5y \end{pmatrix}$.

It is clear that the composition of any two scalings commutes, that is, if $S_{a,b}, S_{a',b'} : \mathbb{R}^2 \to \mathbb{R}^2$ are scalings, then $S_{a,b} \circ S_{a',b'} = S_{a',b'} \circ S_{a,b}$. For

$$(S_{a,b} \circ S_{a',b'})(\mathbf{r}) = S_{a,b}(S_{a',b'}(\mathbf{r})) = S_{a,b}\left(\begin{pmatrix} a'x \\ b'y \end{pmatrix}\right) = \begin{pmatrix} a(a'x) \\ b(b'y) \end{pmatrix},$$

and

$$(S_{a',b'} \circ S_{a,b})(\mathbf{r}) = S_{a',b'}(S_{a,b}(\mathbf{r})) = S_{a',b'}\left(\begin{pmatrix} ax \\ by \end{pmatrix}\right) = \begin{pmatrix} a'(ax) \\ b'(by) \end{pmatrix},$$

from where the commutativity claim is deduced.

Translations and scalings do not necessarily commute, however. For consider the translation $T_{\overrightarrow{\mathbf{i}}}(\mathbf{a}) = \mathbf{a} + \overrightarrow{\mathbf{i}}$ and the scaling $S_{2,1}(\mathbf{a}) = \begin{pmatrix} 2a_1 \\ a_2 \end{pmatrix}$. Then

$$(T_{\overrightarrow{\mathbf{i}}} \circ S_{2,1})\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = T_{\overrightarrow{\mathbf{i}}}\left(S\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)\right) = T_{\overrightarrow{\mathbf{i}}}\left(\begin{pmatrix} -2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

but

$$(S_{2,1} \circ T_{\overrightarrow{\mathbf{i}}})\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = S_{2,1}\left(T_{\overrightarrow{\mathbf{i}}}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)\right) = S_{2,1}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

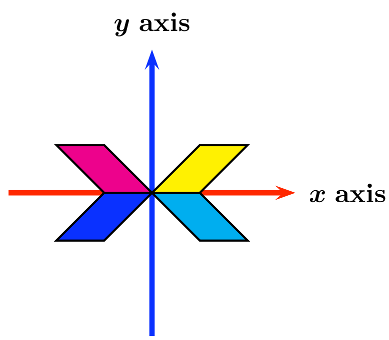

Figure 1.26: Reflexions. The original object (in the first quadrant) is yellow. Its reflexion about the y-axis is magenta (on the second quadrant). Its reflexion about the x-axis is cyan (on the fourth quadrant). Its reflexion about the origin is blue (on the third quadrant).

38 Definition A function $R_H : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *reflexion about the y-axis or horizontal reflexion* if it is of the form $R_H(\mathbf{r}) = \begin{pmatrix} -x \\ y \end{pmatrix}$. A function $R_V : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *reflexion about the x-axis or vertical reflexion* if it is of the form $R_V(\mathbf{r}) = \begin{pmatrix} x \\ -y \end{pmatrix}$. A function $R_O : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *reflexion about origin* if it is of the form $R_H(\mathbf{r}) = \begin{pmatrix} -x \\ -y \end{pmatrix}$.

Some reflexions appear in figure 1.26.

A few short computations establish various commutativity properties among reflexions, translations, and scalings. See problem 1.4.4.

We now define rotations. This definition will be somewhat harder than the others, so let us develop some ancillary results.

Consider a point $\mathbf{r}$ with polar coordinates $x = \rho\cos\alpha$, $y = \rho\sin\alpha$ as in figure 1.27. Here $\rho = \sqrt{x^2 + y^2}$ and $\alpha \in [0; 2\pi[$. If we rotate it, in the levogyrate sense, by an angle θ, we land on the new point $\mathbf{x}'$ with $x' = \rho\cos(\alpha+\theta)$ and $y' = \rho\sin(\alpha+\theta)$. But

$$\rho\cos(\alpha+\theta) = \rho\cos\theta\cos\alpha - \rho\sin\theta\sin\alpha = x\cos\theta - y\sin\theta,$$

and

$$\rho\sin(\alpha+\theta) = \rho\sin\alpha\cos\theta + \rho\sin\theta\cos\alpha = y\cos\theta + x\sin\theta.$$

Hence the point $\begin{pmatrix} x \\ y \end{pmatrix}$ is mapped to the point $\begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$.

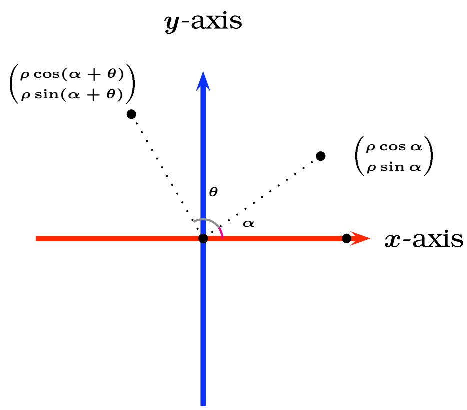

Figure 1.27: Rotation by an angle θ in the levogyrate (counterclockwise) sense from the x-axis.

We may now formulate the definition of a rotation.

39 Definition A function $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *levogyrate rotation about the origin by the angle θ measured from the positive x-axis* if $R_\theta(\mathbf{r}) = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$. Here $\rho = \sqrt{x^2+y^2}$.

Various properties of the composition of rotations with other plane transformations are explored in problems 1.4.5 and 1.4.6.

We now codify some properties shared by scalings, reflexions, and rotations.

40 Definition A function $L : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a *linear transformation* from $\mathbb{R}^2$ to $\mathbb{R}^2$ if for all points $\mathbf{a}, \mathbf{b}$ on the plane and every scalar λ, it is verified that

$$L(\mathbf{a}+\mathbf{b}) = L(\mathbf{a}) + L(\mathbf{b}), \qquad L(\lambda\mathbf{a}) = \lambda L(\mathbf{a}).$$

It is easy to prove that scalings, reflexions, and rotations are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, but not so translations.

41 Definition A function $A : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be an *affine transformation* from $\mathbb{R}^2$ to $\mathbb{R}^2$ if there exists a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ and a fixed vector $\overrightarrow{\mathbf{v}} \in \mathbb{R}^2$ such that for all points $\mathbf{x} \in \mathbb{R}^2$ it is verified that

$$A(\mathbf{x}) = L(\mathbf{x}) + \overrightarrow{\mathbf{v}}.$$

It is easy to see that translations are then affine transformations, where for the linear transformation L in the definition we may take $I : \mathbb{R}^2 \to \mathbb{R}^2$, the identity transformation $I(\mathbf{x}) = \mathbf{x}$.

We have seen that scalings, reflexions and rotations are linear transformations. If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then

$$L(\mathbf{r}) = L(x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}}) = xL(\overrightarrow{\mathbf{i}}) + yL(\overrightarrow{\mathbf{j}}),$$

and thus a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is solely determined by the values $L(\overrightarrow{i})$ and $L(\overrightarrow{j})$. We will now introduce a way to codify these values.

42 Definition Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. The *matrix A_L associated to L* is the 2×2, (2 rows, 2 columns) array whose columns are (in this order) $L\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$ and $L\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$.

43 Example (Scaling Matrices) Let $a > 0, b > 0$ be a real numbers. The matrix of the scaling transformation $S_{a,b}$ is $\begin{bmatrix}a & 0\\0 & b\end{bmatrix}$. For

$$S_{a,b}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}a \cdot 1\\b \cdot 0\end{pmatrix} = \begin{pmatrix}a\\0\end{pmatrix}$$

and

$$S_{a,b}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}0 \cdot 1\\b \cdot 1\end{pmatrix} = \begin{pmatrix}0\\b\end{pmatrix}.$$

44 Example (Reflexion Matrices) It is easy to verify that the matrix for the transformation R_H is $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$, that the matrix for the transformation R_V is $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$, and the matrix for the transformation R_O is $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$.

45 Example (Rotating Matrices) It is easy to verify that the matrix for a rotation R_θ is $\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$.

46 Example (Identity Matrix) The matrix for the identity linear transformation $\mathbf{Id} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{Id}(\mathbf{x}) = \mathbf{x}$ is $\mathbf{I}_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

47 Example (Zero Matrix) The matrix for the null linear transformation $\mathbf{N} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{N}(\mathbf{x}) = \mathbf{O}$ is $\mathbf{0}_2 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$.

From problem 1.4.7 we know that the composition of two linear transformations is also linear. We are now interested in how to codify the matrix of a composition of linear transformations $L_1 \circ L_1$ in terms of their individual matrices.

48 Theorem Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix representation $A_L = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ and let $L' : \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix representation $A_{L'} = \begin{bmatrix}r & s\\t & u\end{bmatrix}$. Then the composition $L \circ L'$ has matrix representation

$$\begin{bmatrix}ar + bt & as + bu\\cr + dt & cs + du\end{bmatrix}.$$

Proof: *We need to find* $(L \circ L')\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$ *and* $(L \circ L')\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$.

We have

$$(L \circ L')\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = L\left(L'\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\right) = L\left(\begin{pmatrix}r\\t\end{pmatrix}\right) = rL(\overrightarrow{\mathrm{i}}) + tL(\overrightarrow{\mathrm{j}}) = r\begin{pmatrix}a\\c\end{pmatrix} + t\begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}ar+bt\\cr+dt\end{pmatrix},$$

and

$$(L \circ L')\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = L\left(L'\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\right) = L\left(\begin{pmatrix}s\\u\end{pmatrix}\right) = sL(\overrightarrow{\mathrm{i}}) + uL(\overrightarrow{\mathrm{j}}) = s\begin{pmatrix}a\\c\end{pmatrix} + u\begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}as+bu\\cs+du\end{pmatrix},$$

whence we conclude that the matrix of $L \circ L'$ *is* $\begin{bmatrix}ar+bt & as+bu\\cr+dt & cs+du\end{bmatrix}$, *as we wanted to shew.*❑

The above motivates the following definition.

49 Definition Let $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ and $B = \begin{bmatrix}r & s\\t & u\end{bmatrix}$ be two 2×2 matrices, and $\lambda \in \mathbb{R}$ be a scalar. We define *matrix addition* as

$$A + B = \begin{bmatrix}a & b\\c & d\end{bmatrix} + \begin{bmatrix}r & s\\t & u\end{bmatrix} = \begin{bmatrix}a+r & b+s\\c+t & d+u\end{bmatrix}.$$

We define *matrix multiplication* as

$$AB = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}r & s\\t & u\end{bmatrix} = \begin{bmatrix}ar+bt & as+bu\\cr+dt & cs+du\end{bmatrix}.$$

We define *scalar multiplication of a matrix* as

$$\lambda A = \lambda\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}\lambda a & \lambda b\\\lambda c & \lambda d\end{bmatrix}.$$

☞ *Since the composition of functions is not necessarily commutative, neither is matrix multiplication. Since the composition of functions is associative, so is matrix multiplication.*

50 Example Let

$$M = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}, \qquad N = \begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}.$$

Then

$$M + N = \begin{bmatrix}2 & 1\\-2 & 2\end{bmatrix}, \qquad 2M = \begin{bmatrix}2 & -2\\0 & 2\end{bmatrix}, \qquad MN = \begin{bmatrix}3 & 1\\-2 & 1\end{bmatrix}.$$

51 Example Find a 2×2 matrix that will transform the square in figure 1.28 into the parallelogram in figure 1.29. Assume in each case that the vertices of the figures are lattice points, that is, coordinate points with integer coordinates.

Solution: ► *Let* $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ *be the desired matrix. Then since*

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

the point $\begin{pmatrix}0\\0\end{pmatrix}$ *is a* fortiori, *transformed to itself. We now assume, without loss of generality, that each vertex of the square is transformed in the same order, counterclockwise, to each vertex of the rectangle. Then*

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}\implies\begin{bmatrix}a\\c\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}\implies a=c=2.$$

Using these values,

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}\implies\begin{bmatrix}a+b\\c+d\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}\implies b=-1,\quad d=1.$$

And so the desired matrix is

$$\begin{bmatrix}2 & -1\\2 & 1\end{bmatrix}.$$

◀

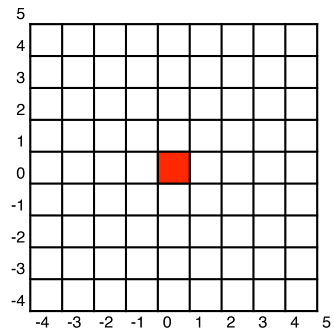

Figure 1.28: Example 51.

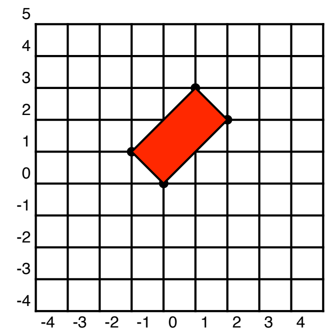

Figure 1.29: Example 51.

Homework

Problem 1.4.1 If $A=\begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}$, $B=\begin{bmatrix}a & b\\1 & -2\end{bmatrix}$ and

$$(A+B)^2=A^2+2AB+B^2,$$

find a and b.

Problem 1.4.2 Consider $\triangle ABC$ with $A=\begin{pmatrix}-1\\2\end{pmatrix}$, $B=\begin{pmatrix}0\\-2\end{pmatrix}$, $C=\begin{pmatrix}2\\1\end{pmatrix}$, as in figure 1.30. Determine the the effects of the following scaling transformations on the triangle: $S_{2,1}$, $S_{1,2}$, and $S_{2,2}$.

Problem 1.4.3 Find the effects of the reflexions $R_{\frac{\pi}{2}}$, $R_{\frac{\pi}{4}}$, $R_{-\frac{\pi}{2}}$, and $R_{-\frac{\pi}{4}}$ on the triangle in figure 1.30.

Problem 1.4.4 Prove that the composition of two reflexions is commutative. Prove that the composition of a reflexion and a scaling is commutative. Prove that the composition of a reflexion and a translation is not necessarily commutative.

Problem 1.4.5 Prove that the composition of two rotations on the plane R_θ and $R_{\theta'}$ satisfies

$$R_\theta\circ R_{\theta'}=R_{\theta+\theta'}=R_{\theta'}\circ R_\theta,$$

and so the composition of two rotations on the plane is commutative.

Problem 1.4.6 Prove that the composition of a scaling and a rotation is not necessarily commutative. Prove that the composition of a rotation and a translation is not necessarily commutative. Prove that the composition of a reflexion and a rotation is not necessarily commutative.

Problem 1.4.7 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ and $L' : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations. Prove that their composition $L \circ L'$ is also a linear transformation.

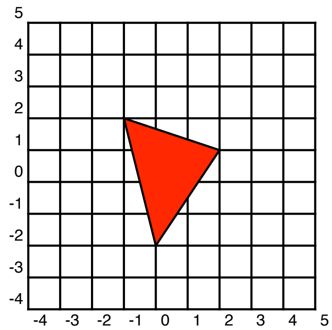

Figure 1.30: Problems 1.4.2, 1.4.8, and 1.4.3.

Problem 1.4.8 Find the effects of the reflexions R_H, R_V, and R_O on the triangle in figure 1.30.

Problem 1.4.9 Find all matrices $A \in \mathbf{M}_{2\times 2}(\mathbb{R})$ such that $A^2 = 0_2$

Problem 1.4.10 Find the image of the figure below (consisting of two circles and a triangle) under the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

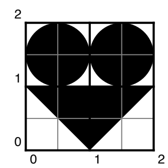

Figure 1.31: Problem 1.4.10.

1.5 Determinants in two dimensions

We now desire to define a way of determining areas of plane figures on the plane. It seems reasonable to require that this area determination agrees with common formulæ of areas of plane figures, in particular, the area of a parallelogram should be as we learn in elementary geometry and the area of a unit square should be 1.

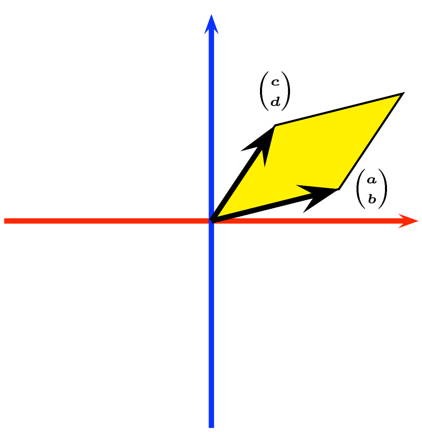

Figure 1.32: Area of a parallelogram.

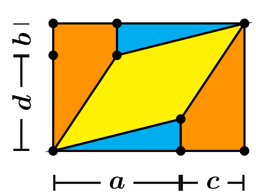

Figure 1.33: $(a + c)(b + d) - 2 \cdot \frac{ab}{2} - 2 \cdot \frac{c(2b+d)}{2} = ad - bc$.

From figures (1.32) and (1.33), the area of a parallelogram spanned by $\begin{bmatrix} a \\ b \end{bmatrix}$, and $\begin{bmatrix} c \\ d \end{bmatrix}$ is

$$D\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = ad - bc.$$

This motivates the following definition.

52 Definition The *determinant* of the 2×2 matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc.$$

Consider now a simple quadrilateral with vertices $r_1 = (x_1, y_1)$, $r_2 = (x_2, y_2)$, $r_3 = (x_3, y_3)$, $r_4 = (x_4, y_4)$, listed in counterclockwise order, as in figure 1.34. This quadrilateral is spanned by the vectors

$$\overrightarrow{r_1 r_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}, \qquad \overrightarrow{r_1 r_4} = \begin{bmatrix} x_4 - x_1 \\ y_4 - y_1 \end{bmatrix},$$

and hence, its area is given by

$$A = \det \begin{bmatrix} x_2 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_4 - y_1 \end{bmatrix} = D(\overrightarrow{r_2} - \overrightarrow{r_1}, \overrightarrow{r_4} - \overrightarrow{r_1}).$$

Similarly, noticing that the quadrilateral is also spanned by

$$\overrightarrow{r_3 r_4} = \begin{bmatrix} x_4 - x_3 \\ y_4 - y_3 \end{bmatrix}, \qquad \overrightarrow{r_3 r_2} = \begin{bmatrix} x_2 - x_3 \\ y_2 - y_3 \end{bmatrix},$$

its area is also given by

$$A = \det \begin{bmatrix} x_4 - x_3 & x_2 - x_3 \\ y_4 - y_3 & y_2 - y_3 \end{bmatrix} = D(\overrightarrow{r_4} - \overrightarrow{r_3}, \overrightarrow{r_2} - \overrightarrow{r_3}).$$

Using the properties derived in Theorem **??**, we see that

$$\begin{aligned}
A &= \tfrac{1}{2}\left(D(\overrightarrow{r_2} - \overrightarrow{r_1}, \overrightarrow{r_4} - \overrightarrow{r_1}) + D(\overrightarrow{r_4} - \overrightarrow{r_3}, \overrightarrow{r_2} - \overrightarrow{r_3})\right) \\
&= \tfrac{1}{2}\left(D(\overrightarrow{r_2}, \overrightarrow{r_4}) - D(\overrightarrow{r_2}, \overrightarrow{r_1}) - D(\overrightarrow{r_1}, \overrightarrow{r_4}) + D(\overrightarrow{r_1}, \overrightarrow{r_1})\right) \\
&\quad + \tfrac{1}{2}\left(D(\overrightarrow{r_4}, \overrightarrow{r_2}) - D(\overrightarrow{r_3}, \overrightarrow{r_2}) - D(\overrightarrow{r_4}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_3})\right) \\
&= \tfrac{1}{2}\left(D(\overrightarrow{r_2}, \overrightarrow{r_4}) - D(\overrightarrow{r_2}, \overrightarrow{r_1}) - D(\overrightarrow{r_1}, \overrightarrow{r_4})\right) + \tfrac{1}{2}\left(D(\overrightarrow{r_4}, \overrightarrow{r_2}) - D(\overrightarrow{r_3}, \overrightarrow{r_2}) - D(\overrightarrow{r_4}, \overrightarrow{r_3})\right) \\
&= \tfrac{1}{2}\left(D(\overrightarrow{r_1}, \overrightarrow{r_2}) + D(\overrightarrow{r_2}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_4}) + D(\overrightarrow{r_4}, \overrightarrow{r_1})\right).
\end{aligned}$$

We conclude that the area of a quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), listed in counterclockwise order is

$$\frac{1}{2}\left(\det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} + \det \begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} + \det \begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \end{bmatrix} + \det \begin{bmatrix} x_4 & x_1 \\ y_4 & y_1 \end{bmatrix}\right). \tag{1.11}$$

Similarly, to find the area of a triangle of vertices $\overrightarrow{r_1} = (x_1, y_1)$, $\overrightarrow{r_2} = (x_2, y_2)$, $\overrightarrow{r_3} = (x_3, y_3)$, listed in counterclockwise order, as in figure 1.35, reflect it about one of its sides, as in figure 1.36, creating a parallelogram. The area of the triangle is now half the area of the parallelogram, which, by virtue of 1.11, is

$$\frac{1}{4}\left(D(\overrightarrow{r_1}, \overrightarrow{r_2}) + D(\overrightarrow{r_2}, \overrightarrow{r}) + D(\overrightarrow{r}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_1})\right).$$

This is equivalent to

$$\frac{1}{2}\left(D(\overrightarrow{r_1}, \overrightarrow{r_2}) + D(\overrightarrow{r_2}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_1})\right) - \frac{1}{4}\left(D(\overrightarrow{r_1}, \overrightarrow{r_2}) - D(\overrightarrow{r_2}, \overrightarrow{r}) - D(\overrightarrow{r}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_1}) + 2D(\overrightarrow{r_2}, \overrightarrow{r_3})\right).$$

We will prove that

$$D(\overrightarrow{r_1}, \overrightarrow{r_2}) - D(\overrightarrow{r_2}, \overrightarrow{r}) - D(\overrightarrow{r}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_1}) + 2D(\overrightarrow{r_2}, \overrightarrow{r_3}) = 0.$$

To do this, we appeal once again to the bi-linearity properties derived in Theorem **??**, and observe, that since we have a parallelogram, $\overrightarrow{r} - \overrightarrow{r_3} = \overrightarrow{r_2} - \overrightarrow{r_1}$, which means $\overrightarrow{r} = \overrightarrow{r_3} + \overrightarrow{r_2} - \overrightarrow{r_1}$. Thus

$$\begin{aligned}
D(\overrightarrow{r_1}, \overrightarrow{r_2}) - D(\overrightarrow{r_2}, \overrightarrow{r}) - D(\overrightarrow{r}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_1}) + 2D(\overrightarrow{r_2}, \overrightarrow{r_3}) &= D(\overrightarrow{r_1}, \overrightarrow{r_2}) - D(\overrightarrow{r_2}, \overrightarrow{r_3} + \overrightarrow{r_2} - \overrightarrow{r_1}) + 2D(\overrightarrow{r_2}, \overrightarrow{r_3}) \\
&\quad -D(\overrightarrow{r_3} + \overrightarrow{r_2} - \overrightarrow{r_1}, \overrightarrow{r_3}) + D(\overrightarrow{r_3}, \overrightarrow{r_1}) \\
&= D(\overrightarrow{r_1}, \overrightarrow{r_2} - \overrightarrow{r_3}) + D(\overrightarrow{r_3} + \overrightarrow{r_2} - \overrightarrow{r_1}, \overrightarrow{r_2} - \overrightarrow{r_3}) \\
&\quad +2D(\overrightarrow{r_2}, \overrightarrow{r_3}) \\
&= D(\overrightarrow{r_3} + \overrightarrow{r_2}, \overrightarrow{r_2} - \overrightarrow{r_3}) + 2D(\overrightarrow{r_2}, \overrightarrow{r_3}) \\
&= D(\overrightarrow{r_3}, \overrightarrow{r_2}) - D(\overrightarrow{r_2}, \overrightarrow{r_3}) + 2D(\overrightarrow{r_2}, \overrightarrow{r_3}) \\
&= D(\overrightarrow{r_3}, \overrightarrow{r_2}) - D(\overrightarrow{r_2}, \overrightarrow{r_3}) \\
&= 0,
\end{aligned}$$

as claimed. We have proved then that the area of a triangle, whose vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) are listed in counterclockwise order, is

$$\frac{1}{2}\left(\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} + \det\begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} + \det\begin{bmatrix} x_3 & x_1 \\ y_3 & y_1 \end{bmatrix}\right). \tag{1.12}$$

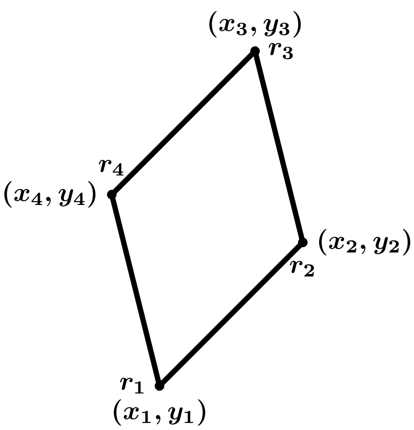

Figure 1.34: Area of a quadrilateral.

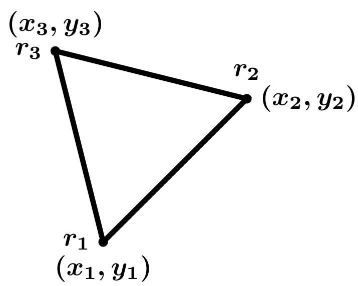

Figure 1.35: Area of a triangle.

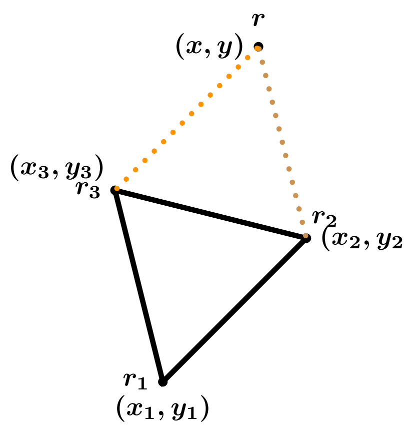

Figure 1.36: Area of a triangle.

In general, we have the following theorem.

53 Theorem (Surveyor's Theorem) Let (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) be the vertices of a simple (non-crossing) polygon, listed in counterclockwise order. Then its area is given by

$$\frac{1}{2}\left(\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} + \det\begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} + \cdots + \det\begin{bmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{bmatrix} + \det\begin{bmatrix} x_n & x_1 \\ y_n & y_1 \end{bmatrix}\right).$$

Proof: *The proof is by induction on n. We have already proved the cases $n = 3$ and $n = 4$ in (1.12) and (1.11), respectively. Consider now a simple polygon P with n vertices. If P is convex then we may take any vertex and draw a line to the other vertices, triangulating the polygon, creating $n - 2$ triangles. If P is not convex, then there must be a vertex that has a reflex angle. A ray produced from this vertex must hit another vertex, creating a diagonal, otherwise the polygon would have infinite area. This diagonal divides the polygon into two sub-polygons. These two sub-polygons are either both convex or at least one is not convex. In the latter case,* *we repeat the argument, finding another diagonal and creating a new sub-polygon. Eventually, since the number of vertices is infinite, we end up triangulating the polygon. Moreover, the polygon can be triangulated in such a way that all triangles inherit the positive orientation of the original polygon but each neighbouring pair of triangles have opposite orientations. Applying (1.12) we obtain that the area is*

$$\sum \det \begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix},$$

where the sum is over each oriented edge. Since each diagonal occurs twice, but having opposite orientations, the terms

$$\det \begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix} + \det \begin{bmatrix} x_j & x_i \\ y_j & y_i \end{bmatrix} = 0,$$

disappear from the sum and we are simply left with

$$\frac{1}{2}\left(\det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} + \det \begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} + \cdots + \det \begin{bmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{bmatrix} + \det \begin{bmatrix} x_n & x_1 \\ y_n & y_1 \end{bmatrix}\right).$$

❑

We may use the software Maple™ in order to speed up computations with vectors. Most of the commands we will need are in the `linalg` package. For example, let us define two vectors, $\overrightarrow{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and a matrix $A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let us compute their dot product, find a unit vector in the direction of $\overrightarrow{a}$, and the angle between the vectors. (There must be either a colon or a semicolon at the end of each statement. The result will not display if a colon is chosen.)

```
> with(linalg):
> a:=vector([2,1]);
```

$$a := [2, 1]$$

```
> b:=vector([1,2]);
```

$$b := [1, 2]$$

```
> normalize(a);
```

$$\left[\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right]$$

```
> dotprod(a,b);
```

$$4$$

```
> angle(a,b);
```

$$\arccos\left(\frac{4}{5}\right)$$

```
> A:=matrix([[1,2],[3,4]]);
```

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

```
> det(A);
```

$$-2$$

Homework

Problem 1.5.1 Find all vectors $\overrightarrow{a} \in \mathbb{R}^2$ such that $\overrightarrow{a} \perp \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $||a|| = \sqrt{13}$.

Problem 1.5.2 (Pythagorean Theorem) If $\overrightarrow{a} \perp \overrightarrow{b}$, prove that

$$||\overrightarrow{a} + \overrightarrow{b}||^2 = \left|\left|\overrightarrow{a}\right|\right|^2 + \left|\left|\overrightarrow{b}\right|\right|^2.$$

Problem 1.5.3 Let a, b be arbitrary real numbers. Prove that

$$(a^2 + b^2)^2 \leq 2(a^4 + b^4).$$

Problem 1.5.4 Let $\overrightarrow{a}$, $\overrightarrow{b}$ be fixed vectors in $\mathbb{R}^2$. Prove that if

$$\forall \overrightarrow{v} \in \mathbb{R}^2, \overrightarrow{v} \bullet \overrightarrow{a} = \overrightarrow{v} \bullet \overrightarrow{b},$$

then $\overrightarrow{a} = \overrightarrow{b}$.

Problem 1.5.5 (Polarisation Identity) Let $\overrightarrow{u}$, $\overrightarrow{v}$ be vectors in $\mathbb{R}^2$. Prove that

$$\overrightarrow{u} \bullet \overrightarrow{v} = \frac{1}{4}\left(||\overrightarrow{u} + \overrightarrow{v}||^2 - ||\overrightarrow{u} - \overrightarrow{v}||^2\right).$$

Problem 1.5.6 Consider two lines on the plane L_1 and L_2 with Cartesian equations $L_1 : y = m_1x + b_1$ and $L_2 : y = m_2x + b_1$, where $m_1 \neq 0$, $m_2 \neq 0$. Using Corollary 22, prove that $L_1 \perp L_2 \iff m_1m_2 = -1$. .

Problem 1.5.7 Find the Cartesian equation of all lines L' passing through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and making an angle of $\frac{\pi}{6}$ radians with the Cartesian line $L : x + y = 1$.

Problem 1.5.8 Let $\overrightarrow{v}$, $\overrightarrow{w}$, be vectors on the plane, with $\overrightarrow{w} \neq \overrightarrow{0}$. Prove that the vector $\overrightarrow{a} = \overrightarrow{v} - \frac{\overrightarrow{v} \bullet \overrightarrow{w}}{||\overrightarrow{w}||^2}\overrightarrow{w}$ is perpendicular to $\overrightarrow{w}$.

1.6 Parametric Curves on the Plane

54 Definition Let $[a; b] \subseteq \mathbb{R}$. A *parametric curve* representation $\mathbf{r}$ of a curve Γ is a function $\mathbf{r} : [a; b] \to \mathbb{R}^2$, with

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

and such that $\mathbf{r}([a; b]) = \Gamma$. $\mathbf{r}(a)$ is the *initial point* of the curve and $\mathbf{r}(b)$ its *terminal point*. A curve is *closed* if its initial point and its final point coincide. The *trace* of the curve $\mathbf{r}$ is the set of all images of $\mathbf{r}$, that is, Γ. If there exist $t_1 \neq t_2$ such that $\mathbf{r}(t_1) = \mathbf{r}(t_2) = \mathbf{p}$, then $\mathbf{p}$ is a *multiple point* of the curve. The curve is *simple* if its has no multiple points. A closed curve whose only multiple points are its endpoints is called a *Jordan curve*.

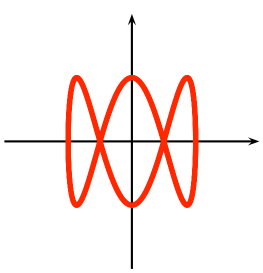

Figure 1.37: $x = \sin 2t$, $y = \cos 6t$.

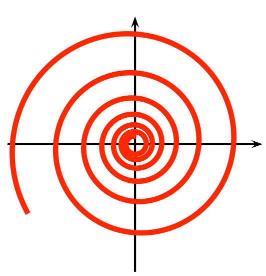

Figure 1.38: $x = 2^{t/10}\cos t$, $y = 2^{t/10}\sin t$.

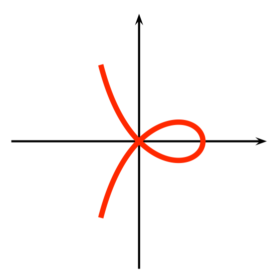

Figure 1.39: $x = \dfrac{1-t^2}{1+t^2}$, $y = \dfrac{t-t^3}{1+t^2}$.

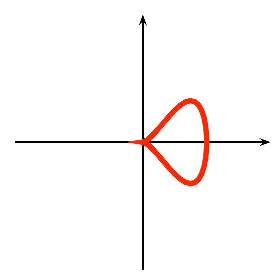

Figure 1.40: $x = (1 + \cos t)/2$, $y = (\sin t)(1 + \cos t)/2$.

Graphing parametric equations is a difficult art, and a theory akin to the one studied for Cartesian equations in a first Calculus course has been developed. Our interest is not in graphing curves, but in obtaining suitable parametrisations of simple Cartesian curves. We mention in passing however that Maple has excellent capabilities for graphing parametric equations. For example, the commands to graph the various curves in figures 1.37 through 1.40 follow.

```
> with(plots):
> plot([sin(2*t),cos(6*t),t=0..2*Pi],x=-5..5,y=-5..5);
> plot([2^(t/10)*cos(t),2^(t/10)*sin(t),t=-20..10],x=-5..5,y=-5..5);
> plot([(1-t^2)/(1+t^2),(t-t^3)/(1+t^2),t=-2..2],x=-5..5,y=-5..5);
> plot([(1+cos(t))/2,sin(t)*(1+cos(t))/2,t=0..2*Pi],x=-5..5,y=-5..5);
```

Our main focus of attention will be the following. Given a Cartesian curve with equation $f(x, y) = 0$, we wish to find suitable parametrisations for them. That is, we want to find functions $x : t \mapsto a(t)$, $y : t \mapsto b(t)$ and an interval I such that the graphs of $f(x, y) = 0$ and $f(a(t), b(t)) = 0$, $t \in I$ coincide. These parametrisations may differ in features, according to the choice of functions and the choice of intervals.

55 Example Consider the parabola with Cartesian equation $y = x^2$. We will give various parametrisations for portions of this curve.

1. If $x = t$ and $y = t^2$, then clearly $y = t^2 = x^2$. This works for every $t \in \mathbb{R}$, and hence the parametrisation
$$x = t, \quad y = t^2, \quad t \in \mathbb{R}$$
works for the whole curve. Notice that as t increases, the curve is traversed from left to right.

2. If $x = \sqrt{t}$ and $y = t$, then again $y = t = (\sqrt{t})^2 = x^2$. This works only for $t \geq 0$, and hence the parametrisation
$$x = \sqrt{t}, \quad y = t, \quad t \in [0; +\infty[$$
gives the half of the curve for which $x \geq 0$. As t increases, the curve is traversed from left to right.

3. Similarly, if $x = -\sqrt{t}$ and $y = t$, then again $y = t = (-\sqrt{t})^2 = x^2$. This works only for $t \geq 0$, and hence the parametrisation
$$x = -\sqrt{t}, \quad y = t, \quad t \in [0; +\infty[$$
gives the half of the curve for which $x \leq 0$. As t increases, x decreases, and so the curve is traversed from right to left.

4. If $x = \cos t$ and $y = \cos^2 t$, then again $y = \cos^2 t = (\cos t)^2 = x^2$. Both x and y are periodic with period 2π, and so this parametrisation only agrees with the curve $y = x^2$ when $-1 \leq x \leq 1$. For $t \in [0; \pi]$, the cosine decreases from 1 to -1 and so the curve is traversed from right to left in this interval.

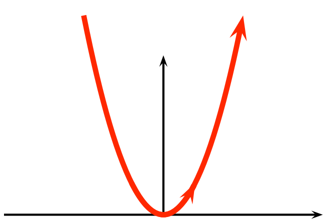

Figure 1.41: $x = t, y = t^2$, $t \in \mathbb{R}$.

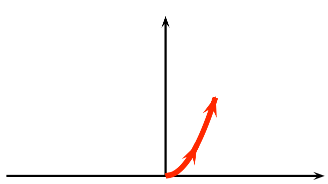

Figure 1.42: $x = \sqrt{t}, y = t$, $t \in [0; +\infty[$.

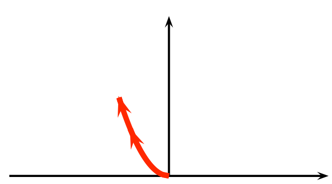

Figure 1.43: $x = -\sqrt{t}, y = t$, $t \in [0; +\infty[$.

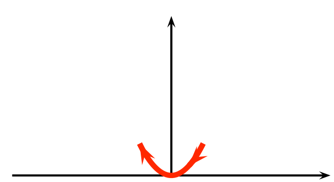

Figure 1.44: $x = \cos t, y = \cos^2 t$, $t \in [0; \pi]$.

The identities
$$\cos^2\theta + \sin^2\theta = 1, \qquad \tan^2\theta - \sec^2\theta = 1, \qquad \cosh^2\theta - \sinh^2\theta = 1,$$
are often useful when parametrising quadratic curves.

56 Example Give two distinct parametrisations of the ellipse $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$.

1. The first parametrisation must satisfy that as t traverses the values in the interval $[0; 2\pi]$, one starts at the point $(3, -2)$, traverses the ellipse once counterclockwise, finishing at $(3, -2)$.

2. The second parametrisation must satisfy that as t traverses the interval $[0; 1]$, one starts at the point $(3, -2)$, traverses the ellipse twice clockwise, and returns to $(3, -2)$.

Solution: ▶ *What formula do we know where a sum of two squares equals* 1*? We use a trigonometric substitution, a sort of "polar coordinates." Observe that for* $t \in [0; 2\pi]$*, the point* $(\cos t, \sin t)$ *traverses the unit circle once, starting at* $(1, 0)$ *and ending there. Put*

$$\frac{x-1}{2} = \cos t \implies x = 1 + 2\cos t,$$

and

$$\frac{y+2}{3} = \sin t \implies y = -2 + 3\sin t.$$

Then

$$x = 1 + 2\cos t, \quad y = -2 + 3\sin t, \quad t \in [0; 2\pi]$$

is the desired first parametrisation.

For the second parametrisation, notice that as t *traverses the interval* $[0; 1]$*,* $(\sin 4\pi t, \cos 4\pi t)$ *traverses the unit interval twice, clockwise, but begins and ends at the point* $(0, 1)$*. To begin at the point* $(1, 0)$ *we must make a shift:* $\left(\sin\left(4\pi t + \frac{\pi}{2}\right), \cos\left(4\pi t + \frac{\pi}{2}\right)\right)$ *will start at* $(1, 0)$ *and travel clockwise twice, as* t *traverses* $[0; 1]$*. Hence we may take*

$$x = 1 + 2\sin\left(4\pi t + \frac{\pi}{2}\right), \quad y = -2 + 3\cos\left(4\pi t + \frac{\pi}{2}\right), \quad t \in [0; 1]$$

as our parametrisation. ◀

Some classic curves can be described by mechanical means, as the curves drawn by a spirograph. We will consider one such curve.

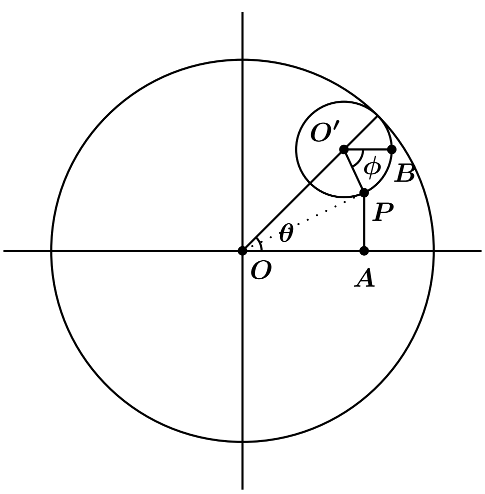

Figure 1.45: Construction of the hypocycloid.

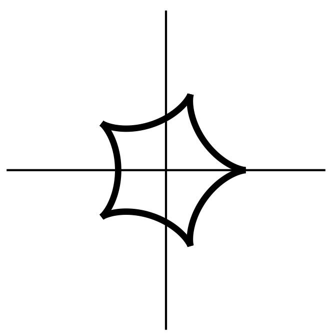

Figure 1.46: Hypocycloid with $R = 5$, $\rho = 1$.

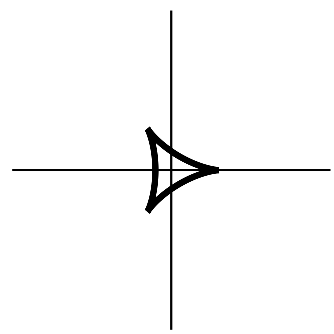

Figure 1.47: Hypocycloid with $R = 3$, $\rho = 2$.

57 Example A *hypocycloid* is a curve traced out by a fixed point P on a circle $\mathscr{C}$ of radius ρ as $\mathscr{C}$ rolls on the inside of a circle with centre at O and radius R. If the initial position of P is $\begin{pmatrix} R \\ 0 \end{pmatrix}$, and θ is the angle, measured counterclockwise, that a ray starting at O and passing through the centre of $\mathscr{C}$ makes with the x-axis, shew that a parametrisation of the hypocycloid is

$$x = (R-\rho)\cos\theta + \rho\cos\left(\frac{(R-\rho)\theta}{\rho}\right),$$

$$y = (R-\rho)\sin\theta - \rho\sin\left(\frac{(R-\rho)\theta}{\rho}\right).$$

Solution: ► *Suppose that starting from $\theta = 0$, the centre O' of the small circle moves counterclockwise inside the larger circle by an angle θ, and the point $P = (x, y)$ moves clockwise an angle ϕ. The arc length travelled by the centre of the small circle is $(R-\rho)\theta$ radians. At the same time the point P has rotated $\rho\phi$ radians, and so $(R-\rho)\theta = \rho\phi$. See figure 1.45, where $O'B$ is parallel to the x-axis.*

Let A be the projection of P on the x-axis. Then $\angle OAP = \angle OPO' = \frac{\pi}{2}$, $\angle OO'P = \pi - \phi - \theta$, $\angle POA = \frac{\pi}{2} - \phi$, and $OP = (R-\rho)\sin(\pi - \phi - \theta)$. Hence

$$x = (OP)\cos\angle POA = (R-\rho)\sin(\pi-\phi-\theta)\cos(\frac{\pi}{2}-\phi),$$

$$y = (R-\rho)\sin(\pi-\phi-\theta)\sin(\frac{\pi}{2}-\phi).$$

Now

$$\begin{aligned} x &= (R-\rho)\sin(\pi-\phi-\theta)\cos(\frac{\pi}{2}-\phi) \\ &= (R-\rho)\sin(\phi+\theta)\sin\phi \\ &= \frac{(R-\rho)}{2}(\cos\theta - \cos(2\phi+\theta)) \\ &= (R-\rho)\cos\theta - \frac{(R-\rho)}{2}(\cos\theta + \cos(2\phi+\theta)) \\ &= (R-\rho)\cos\theta - (R-\rho)(\cos(\theta+\phi)\cos\phi). \end{aligned}$$

Also, $\cos(\theta+\phi) = -\cos(\pi-\theta-\phi) = -\frac{\rho}{OO'} = -\frac{\rho}{R-\rho}$ *and* $\cos\phi = \cos\left(\frac{(R-\rho)\theta}{\rho}\right)$ *and so*

$$x = (R-\rho)\cos\theta - (R-\rho)(\cos(\theta+\phi)\cos\phi = (R-\rho)\cos\theta + \rho\cos\left(\frac{(R-\rho)\theta}{\rho}\right),$$

as required. The identity for y is proved similarly. A particular example appears in figure 1.47. ◄

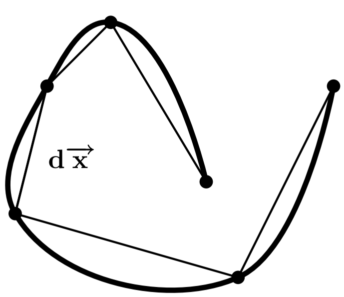

Figure 1.48: Length of a curve.

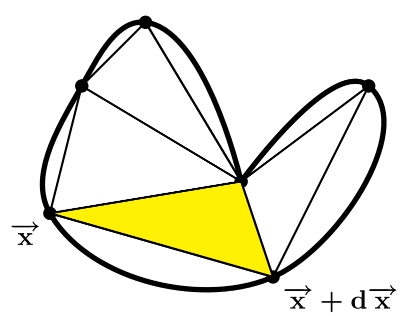

Figure 1.49: Area enclosed by a simple closed curve

Given a curve Γ how can we find its length? The idea, as seen in figure 1.48 is to consider the projections dx, dy at each point. The length of the vector

$$\mathrm{dr} = \begin{bmatrix} dx \\ dy \end{bmatrix}$$

is
$$||\mathbf{dr}|| = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}.$$
Hence the length of Γ is given by
$$\int_\Gamma ||\mathbf{dr}|| = \int_\Gamma \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}. \tag{1.13}$$

Similarly, suppose that Γ is a simple closed curve in $\mathbb{R}^2$. How do we find the (oriented) area of the region it encloses? The idea, borrowed from finding areas of polygons, is to split the region into triangles, each of area
$$\frac{1}{2}\det\begin{bmatrix} x & x+\mathrm{d}x \\ y & y+\mathrm{d}y \end{bmatrix} = \frac{1}{2}\det\begin{bmatrix} x & \mathrm{d}x \\ y & \mathrm{d}y \end{bmatrix} = \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x),$$
and to sum over the closed curve, obtaining a total oriented area of
$$\frac{1}{2}\oint_\Gamma \det\begin{bmatrix} x & \mathrm{d}x \\ y & \mathrm{d}y \end{bmatrix} = \frac{1}{2}\oint_\Gamma (x\mathrm{d}y - y\mathrm{d}x). \tag{1.14}$$
Here $\oint_\Gamma$ denotes integration around the closed curve.

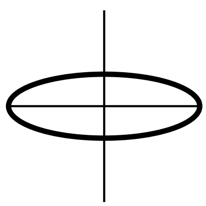

Figure 1.50: Example 58.

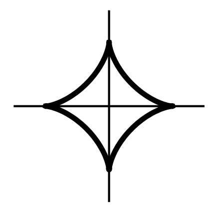

Figure 1.51: Example 59.

58 Example Let $(A, B) \in \mathbb{R}^2, A > 0, B > 0$. Find a parametrisation of the ellipse
$$\Gamma : \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1\right\}.$$
Furthermore, find an integral expression for the perimeter of this ellipse and find the area it encloses.

Solution: ► *Consider the parametrisation* $\Gamma : [0; 2\pi] \to \mathbb{R}^2$, *with*
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A\cos t \\ B\sin t \end{bmatrix}.$$
This is a parametrisation of the ellipse, for
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = \frac{A^2\cos^2 t}{A^2} + \frac{B^2\sin^2 t}{B^2} = \cos^2 t + \sin^2 t = 1.$$
Notice that this parametrisation goes around once the ellipse counterclockwise. The perimeter of the ellipse is given by
$$\int_\Gamma ||\mathrm{d}\,\overrightarrow{\mathbf{r}}|| = \int_0^{2\pi} \sqrt{A^2\sin^2 t + B^2\cos^2 t}\,\mathrm{d}t.$$

The above integral is an elliptic integral, *and we do not have a closed form for it (in terms of the elementary functions studied in Calculus I). We will have better luck with the area of the ellipse, which is given by*

$$\begin{aligned}\frac{1}{2}\oint_{\Gamma}(x\mathrm{d}y - y\mathrm{d}x) &= \frac{1}{2}\oint(A\cos t\,\mathrm{d}(B\sin t) - B\sin t\,\mathrm{d}(A\cos t))\\ &= \frac{1}{2}\int_0^{2\pi}(AB\cos^2 t + AB\sin^2 t))\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^{2\pi} AB\,\mathrm{d}t\\ &= \pi AB.\end{aligned}$$

◀

59 Example Find a parametric representation for the astroid

$$\Gamma : \left\{(x, y) \in \mathbb{R}^2 : x^{2/3} + y^{2/3} = 1\right\},$$

in figure 1.51. Find the perimeter of the astroid and the area it encloses.

Solution: ► *Take*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix}$$

with $t \in [0; 2\pi]$. *Then*

$$x^{2/3} + y^{2/3} = \cos^2 t + \sin^2 t = 1.$$

The perimeter of the astroid is

$$\begin{aligned}\int_{\Gamma}||\mathrm{d}\mathbf{r}|| &= \int_0^{2\pi}\sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}\,\mathrm{d}t\\ &= \int_0^{2\pi} 3|\sin t\cos t|\,\mathrm{d}t\\ &= \frac{3}{2}\int_0^{2\pi}|\sin 2t|\,\mathrm{d}t\\ &= 6\int_0^{\pi/2}\sin 2t\,\mathrm{d}t\\ &= 6.\end{aligned}$$

The area of the astroid is given by

$$\begin{aligned}\frac{1}{2}\oint_{\Gamma}(x\mathrm{d}y - y\mathrm{d}x) &= \frac{1}{2}\oint(\cos^3 t\,\mathrm{d}(\sin^3 t) - \sin^3 t\,\mathrm{d}(\cos^3 t))\\ &= \frac{1}{2}\int_0^{2\pi}(3\cos^4 t\sin^2 t + 3\sin^4 t\cos^2 t))\,\mathrm{d}t\\ &= \frac{3}{2}\int_0^{2\pi}(\sin t\cos t)^2\,\mathrm{d}t\\ &= \frac{3}{8}\int_0^{2\pi}(\sin 2t)^2\,\mathrm{d}t\\ &= \frac{3}{16}\int_0^{2\pi}(1-\cos 4t)\,\mathrm{d}t\\ &= \frac{3\pi}{8}.\end{aligned}$$

We can use Maple™ (at least version 10) to calculate the above integrals. For example, if $(x, y) = (\cos^3 t, \sin^3 t)$, *to compute the arc length we use the path integral command and to compute the area, we use the line integral command with the vector field* $\begin{bmatrix} -y/2 \\ x/2 \end{bmatrix}$.

```
> with(Student[VectorCalculus]):
> PathInt( 1, [x,y]=Path( <(cos(t))^3,(sin(t))^3>,0..2*Pi));
> LineInt( VectorField(<-y/2,x/2>), Path( <(cos(t))^3,(sin(t))^3>,0..2*Pi));
```

◀

We include here for convenience, some Maple commands to compute various arc lengths and areas.

60 Example To obtain the arc length of the path in figure 1.52, we type

```
> with(Student[VectorCalculus]):
> PathInt(1, [x,y] = LineSegments( <0,0>, <1,1>, <1,2> ,<2,1>,<3,3>,<4,1>);
```

To obtain the arc length of the path in figure 1.53, we type

```
> with(Student[VectorCalculus]):
> PathInt(1, [x,y] =  Arc( Circle( <0,0>, 3), Pi/6, Pi/5 ) ;
```

To obtain the area inside the curve in 1.54

```
> with(Student[VectorCalculus]):
> LineInt( VectorField(<-y/2,x/2>),
> Path( <(1+cos(t))*(cos(t))+1,(1+cos(t))*(sin(t))+2>,0..2*Pi));
```

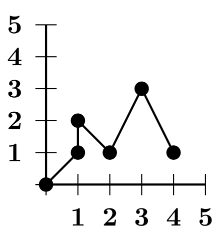

Figure 1.52: Line Path.

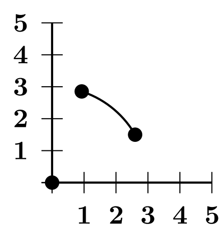

Figure 1.53: Arc of circle of radius 3, angle $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{5}$.

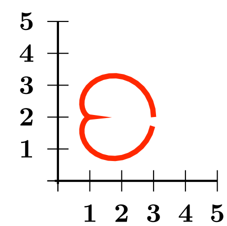

Figure 1.54: $x = 1 + (1 + \cos t)(\cos t), y = 2 + (1 + \cos t)(\sin t)$.

Homework

Problem 1.6.1 A curve is represented parametrically by $x(t) = t^3 - 2t$, $y(t) = t^3 + 2t$. Find its Cartesian equation.

Problem 1.6.2 Give an implicit Cartesian equation for the parametric representation $x = \dfrac{t^2}{1+t^5}$, $y = \dfrac{t^3}{1+t^5}$.

Problem 1.6.3 Let a, b, c, d be strictly positive real constants. In each case give an implicit Cartesian equation for the parametric representation and describe the trace of the parametric curve.

1. $x = at + b$, $y = ct + d$
2. $x = \cos t$, $y = 0$
3. $x = a\cosh t$, $y = b\sinh t$
4. $x = a\sec t$, $y = b\tan t$, $t \in]-\frac{\pi}{2}; \frac{\pi}{2}[$

Problem 1.6.4 Parametrise the curve $y = \log\cos x$ for $0 \leq x \leq \frac{\pi}{3}$. Then find its arc length.

Problem 1.6.5 Describe the trace of the parametric curve

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin t \\ 2\sin t + 1 \end{bmatrix}, \quad t \in [0; 4\pi].$$

Problem 1.6.6 Consider the plane curve defined implicitly by $\sqrt{x} + \sqrt{y} = 1$. Give a suitable parametrisation of this curve, and find its length. The graph of the curve appears in figure 1.55.

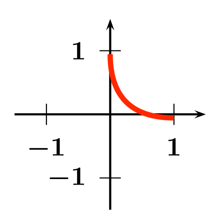

Figure 1.55: Problem 1.6.6.

Problem 1.6.7 Consider the graph given parametrically by $x(t) = t^3 + 1$, $y(t) = 1 - t^2$. Find the area under the graph, over the x axis, and between the lines $x = 1$ and $x = 2$.

Problem 1.6.8 Find the arc length of the curve given parametrically by $x(t) = 3t^2$, $y(t) = 2t^3$ for $0 \leq t \leq 1$.

Problem 1.6.9 Let $\mathscr{C}$ be the curve in $\mathbb{R}^2$ defined by

$$x(t) = \frac{t^2}{2}, \quad y(t) = \frac{(2t+1)^{3/2}}{3}, \quad t \in [-\frac{1}{2}; +\frac{1}{2}].$$

Find the length of this curve.

Problem 1.6.10 Find the area enclosed by the curve $x(t) = \sin^3 t, y(t) = (\cos t)(1 + \sin^2 t)$. The curve appears in figure 1.56.

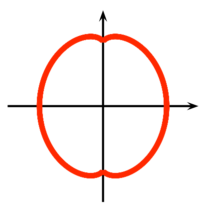

Figure 1.56: Problem 1.6.10.

Problem 1.6.11 Let $\mathscr{C}$ be the curve in $\mathbb{R}^2$ defined by

$$x(t) = \frac{3t}{1+t^3}, \quad y(t) = \frac{3t^2}{1+t^3}, \quad t \in \mathbb{R} \setminus \{-1\},$$

which you may see in figure 1.57. Find the area enclosed by the loop of this curve.

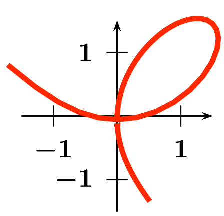

Figure 1.57: Problem 1.6.11.

Problem 1.6.12 Let P be a point at a distance d from the centre of a circle of radius ρ. The curve traced out by P as the circle rolls along a straight line, without slipping, is called a *cycloid*. Find a parametrisation of the cycloid.

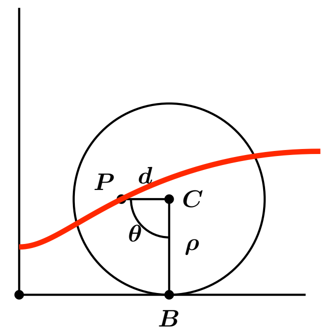

Figure 1.58: Cycloid

Problem 1.6.13 Find the arc length of the arc of the cycloid $x = \rho(t - \cos t)$, $y = \rho(1 - \cos t)$, $t \in [0; 2\pi]$.

Problem 1.6.14 Find the length of the parametric curve given by

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad t \in [0; \pi].$$

Problem 1.6.15 A shell strikes an airplane flying at height h above the ground. It is known that the shell was fired from a gun on the ground with a muzzle velocity of magnitude V, but the position of the gun and its angle of elevation are both unknown. Deduce that the gun is situated within a circle whose centre lies directly below the airplane and whose radius is

$$\frac{V\sqrt{V^2 - 2gh}}{g}.$$

Problem 1.6.16 The parabola $y^2 = -4px$ rolls without slipping around the parabola $y^2 = 4px$. Find the equation of the locus of the vertex of the rolling parabola.

1.7 Vectors in Space

61 Definition The 3-dimensional Cartesian Space is defined and denoted by

$$\mathbb{R}^3 = \{\mathbf{r} = (x, y, z) : x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}.$$

In figure 1.59 we have pictured the point $(2, 1, 3)$.

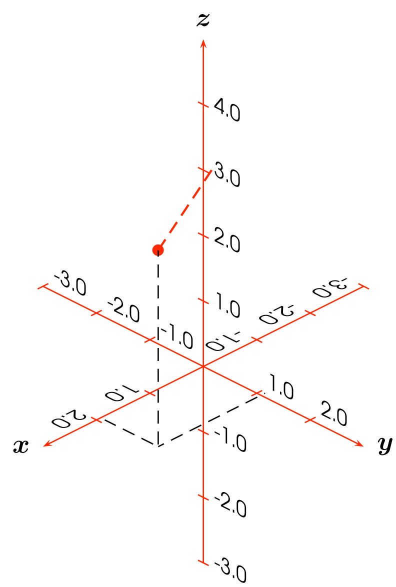

Figure 1.59: A point in space.

Having oriented the z axis upwards, we have a choice for the orientation of the the x and y-axis. We adopt a convention known as a *right-handed coordinate system*, as in figure 1.60. Let us explain. In analogy to $\mathbb{R}^2$ we put

$$\overrightarrow{\mathrm{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \overrightarrow{\mathrm{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \overrightarrow{\mathrm{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and observe that

$$\mathrm{r} = (x, y, z) = x\,\overrightarrow{\mathrm{i}} + y\,\overrightarrow{\mathrm{j}} + z\,\overrightarrow{\mathrm{k}}\,.$$

Most of what we did in $\mathbb{R}^2$ transfers to $\mathbb{R}^3$ without major complications.

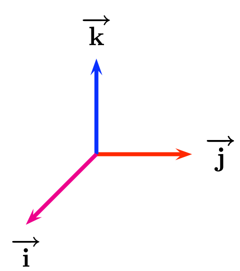

Figure 1.60: Right-handed system.

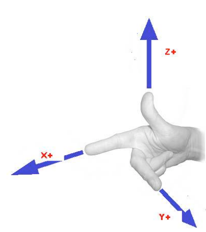

Figure 1.61: Right Hand.

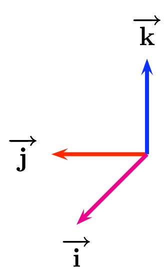

Figure 1.62: Left-handed system.

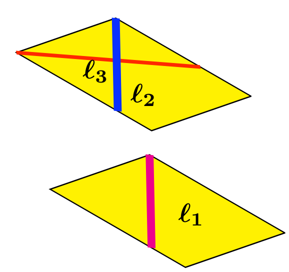

Figure 1.63: $\ell_1 \parallel \ell_2$. ℓ_1 and ℓ_3 are skew.

62 Definition The dot product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in $\mathbb{R}^3$ is

$$\overrightarrow{a}\bullet\overrightarrow{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

The norm of a vector $\overrightarrow{a}$ in $\mathbb{R}^3$ is

$$\left|\left|\overrightarrow{a}\right|\right| = \sqrt{\overrightarrow{a}\bullet\overrightarrow{a}} = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}.$$

Just as in $\mathbb{R}^2$, the dot product satisfies $\overrightarrow{a}\bullet\overrightarrow{b} = \left|\left|\overrightarrow{a}\right|\right|\left|\left|\overrightarrow{b}\right|\right|\cos\theta$, where $\theta \in [0;\pi]$ is the convex angle between the two vectors.

The Cauchy-Schwarz-Bunyakovsky Inequality takes the form

$$|\overrightarrow{a}\bullet\overrightarrow{b}| \leq \left|\left|\overrightarrow{a}\right|\right|\left|\left|\overrightarrow{b}\right|\right| \implies |a_1b_1 + a_2b_2 + a_3b_3| \leq (a_1^2 + a_2^2 + a_3^2)^{1/2}(b_1^2 + b_2^2 + b_3^2)^{1/2},$$

equality holding if an only if the vectors are parallel.

63 Example Let x, y, z be positive real numbers such that $x^2 + 4y^2 + 9z^2 = 27$. Maximise $x + y + z$.

Solution: ▶ *Since x, y, z are positive, $|x + y + z| = x + y + z$. By Cauchy's Inequality,*

$$|x+y+z| = \left|x + 2y\left(\frac{1}{2}\right) + 3z\left(\frac{1}{3}\right)\right| \leq (x^2+4y^2+9z^2)^{1/2}\left(1 + \frac{1}{4} + \frac{1}{9}\right)^{1/2} = \sqrt{27}\left(\frac{7}{6}\right) = \frac{7\sqrt{3}}{2}.$$

Equality occurs if and only if

$$\begin{bmatrix} x \\ 2y \\ 3z \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} \implies x = \lambda, y = \frac{\lambda}{4}, z = \frac{\lambda}{9} \implies \lambda^2 + \frac{\lambda^2}{4} + \frac{\lambda^2}{9} = 27 \implies \lambda = \pm\frac{18\sqrt{3}}{7}.$$

Therefore for a maximum we take

$$x = \frac{18\sqrt{3}}{7}, \quad y = \frac{9\sqrt{3}}{14}, \quad z = \frac{2\sqrt{3}}{7}.$$

◀

64 Definition Let a be a point in $\mathbb{R}^3$ and let $\overrightarrow{v} \neq \overrightarrow{0}$ be a vector in $\mathbb{R}^3$. The *parametric line* passing through a in the direction of $\overrightarrow{v}$ is the set

$$\left\{\mathbf{r} \in \mathbb{R}^3 : \mathbf{r} = \mathbf{a} + t\overrightarrow{v}\right\}.$$

65 Example Find the parametric equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$.

Solution: ▶ *The line follows the direction*

$$\begin{bmatrix} 1-(-2) \\ 2-(-1) \\ 3-0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

The desired equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

◀

Given two lines in space, one of the following three situations might arise: (i) the lines intersect at a point, (ii) the lines are parallel, (iii) the lines are skew (non-parallel, one over the other, without intersecting, lying on different planes). See figure 1.63.

Consider now two non-zero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in $\mathbb{R}^3$. If $\overrightarrow{a} \parallel \overrightarrow{b}$, then the set

$$\{s\overrightarrow{a} + t\overrightarrow{b} : s \in \mathbb{R}, t \in \mathbb{R}\} = \{\lambda\overrightarrow{a} : \lambda \in \mathbb{R}\},$$

which is a line through the origin. Suppose now that $\overrightarrow{a}$ and $\overrightarrow{b}$ are not parallel. We saw in the preceding chapter that if the vectors were on the plane, they would span the whole plane $\mathbb{R}^2$. In the case at hand the vectors are in space, they still span a plane, passing through the origin. Thus

$$\{s\overrightarrow{a} + t\overrightarrow{b} : s \in \mathbb{R}, t \in \mathbb{R}, \overrightarrow{a} \not\parallel \overrightarrow{b}\}$$

is a plane passing through the origin. We will say, abusing language, that two vectors are *coplanar* if there exists bi-point representatives of the vector that lie on the same plane. We will say, again abusing language, that a vector is *parallel to a specific plane* or that it *lies on a specific plane* if there exists a bi-point representative of the vector that lies on the particular plane. All the above gives the following result.

66 Theorem Let $\overrightarrow{v}, \overrightarrow{w}$ in $\mathbb{R}^3$ be non-parallel vectors. Then every vector $\overrightarrow{u}$ of the form

$$\overrightarrow{u} = a\overrightarrow{v} + b\overrightarrow{w},$$

a, b arbitrary scalars, is coplanar with both $\overrightarrow{v}$ and $\overrightarrow{w}$. Conversely, any vector $\overrightarrow{t}$ coplanar with both $\overrightarrow{v}$ and $\overrightarrow{w}$ can be uniquely expressed in the form

$$\overrightarrow{t} = p\overrightarrow{v} + q\overrightarrow{w}.$$

See figure 1.64.

From the above theorem, if a vector $\overrightarrow{a}$ is not a linear combination of two other vectors $\overrightarrow{b}, \overrightarrow{c}$, then linear combinations of these three vectors may lie outside the plane containing $\overrightarrow{b}, \overrightarrow{c}$. This prompts the following theorem.

67 Theorem Three vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ in $\mathbb{R}^3$ are said to be *linearly independent* if

$$\alpha\overrightarrow{a} + \beta\overrightarrow{b} + \gamma\overrightarrow{c} = \overrightarrow{0} \implies \alpha = \beta = \gamma = 0.$$

Any vector in $\mathbb{R}^3$ can be written as a linear combination of three linearly independent vectors in $\mathbb{R}^3$.

A plane is determined by three non-collinear points. Suppose that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are non-collinear points on the same plane and that $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is another arbitrary point on this plane. Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are non-collinear, $\overrightarrow{\mathbf{ab}}$ and $\overrightarrow{\mathbf{ac}}$, which are coplanar, are non-parallel. Since $\overrightarrow{\mathbf{ax}}$ also lies on the plane, we have by Lemma 66, that there exist real numbers p, q with

$$\overrightarrow{\mathbf{ar}} = p\overrightarrow{\mathbf{ab}} + q\overrightarrow{\mathbf{ac}}.$$

By Chasles' Rule,

$$\overrightarrow{r} = \overrightarrow{a} + p(\overrightarrow{b} - \overrightarrow{a}) + q(\overrightarrow{c} - \overrightarrow{a}),$$

is the equation of a plane containing the three non-collinear points a, b, and c, where $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are the position vectors of these points. Thus we have the following theorem.

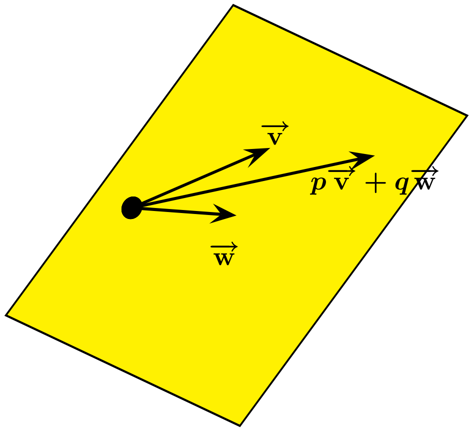

Figure 1.64: Theorem 66.

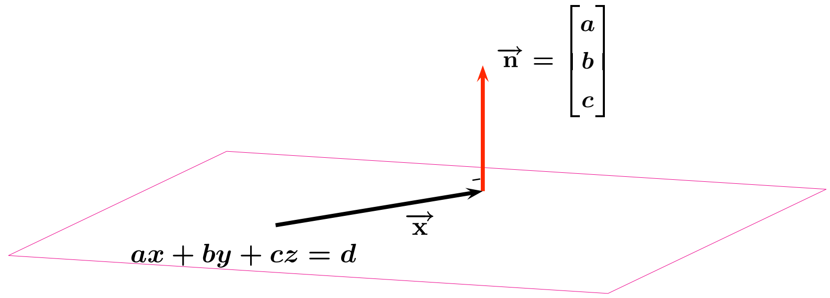

Figure 1.65: Theorem 69.

68 Theorem Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be linearly independent vectors. The *parametric equation* of a plane containing the point a, and parallel to the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is given by

$$\overrightarrow{r}-\overrightarrow{a}=p\overrightarrow{u}+q\overrightarrow{v}.$$

Componentwise this takes the form

$$\begin{aligned}x-a_1&=pu_1+qv_1,\\ y-a_2&=pu_2+qv_2,\\ z-a_3&=pu_3+qv_3.\end{aligned}$$

Multiplying the first equation by $u_2v_3-u_3v_2$, the second by $u_3v_1-u_1v_3$, and the third by $u_1v_2-u_2v_1$, we obtain,

$$\begin{aligned}(u_2v_3-u_3v_2)(x-a_1)&=(u_2v_3-u_3v_2)(pu_1+qv_1),\\ (u_3v_1-u_1v_3)(y-a_2)&=(u_3v_1-u_1v_3)(pu_2+qv_2),\\ (u_1v_2-u_2v_1)(z-a_3)&=(u_1v_2-u_2v_1)(pu_3+qv_3).\end{aligned}$$

Adding gives,

$$(u_2v_3-u_3v_2)(x-a_1)+(u_3v_1-u_1v_3)(y-a_2)+(u_1v_2-u_2v_1)(z-a_3)=0.$$

Put

$$a=u_2v_3-u_3v_2,\qquad b=u_3v_1-u_1v_3,\qquad c=u_1v_2-u_2v_1,$$

and

$$d=a_1(u_2v_3-u_3v_2)+a_2(u_3v_1-u_1v_3)+a_3(u_1v_2-u_2v_1).$$

Since $\overrightarrow{v}$ is linearly independent from $\overrightarrow{u}$, not all of a, b, c are zero. This gives the following theorem.

69 Theorem The equation of the plane in space can be written in the form

$$ax+by+cz=d,$$

which is the *Cartesian equation* of the plane. Here $a^2+b^2+c^2\neq 0$, that is, at least one of the coefficients is non-zero. Moreover, the vector $\overrightarrow{n}=\begin{bmatrix}a\\b\\c\end{bmatrix}$ is normal to the plane with Cartesian equation $ax+by+cz=d$.

Proof: *We have already proved the first statement. For the second statement, observe that if $\overrightarrow{u}$ and $\overrightarrow{v}$ are non-parallel vectors and $\overrightarrow{r} - \overrightarrow{a} = p\overrightarrow{u} + q\overrightarrow{v}$ is the equation of the plane containing the point* a *and parallel to the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, then if $\overrightarrow{n}$ is simultaneously perpendicular to $\overrightarrow{u}$ and $\overrightarrow{v}$ then $(\overrightarrow{r} - \overrightarrow{a})\bullet\overrightarrow{n} = 0$ for $\overrightarrow{u}\bullet\overrightarrow{n} = 0 = \overrightarrow{v}\bullet\overrightarrow{n}$. Now, since at least one of a, b, c is non-zero, we may assume $a \neq 0$. The argument is similar if one of the other letters is non-zero and $a = 0$. In this case we can see that*

$$x = \frac{d}{a} - \frac{b}{a}y - \frac{c}{a}z.$$

Put $y = s$ and $z = t$. Then

$$\begin{pmatrix} x - \dfrac{d}{a} \\ y \\ z \end{pmatrix} = s\begin{bmatrix} -\dfrac{b}{a} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\dfrac{c}{a} \\ 0 \\ 1 \end{bmatrix}$$

is a parametric equation for the plane. We have

$$a\left(-\frac{b}{a}\right) + b\,(1) + c\,(0) = 0, \qquad a\left(-\frac{c}{a}\right) + b\,(0) + c\,(1) = 0,$$

and so $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *is simultaneously perpendicular to* $\begin{bmatrix} -\dfrac{b}{a} \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -\dfrac{c}{a} \\ 0 \\ 1 \end{bmatrix}$*, proving the second statement.*❑

70 Example The equation of the plane passing through the point $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and normal to the vector $\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ is

$$-3(x - 1) + 2(y + 1) + 4(z - 2) = 0 \implies -3x + 2y + 4z = 3.$$

71 Example Find both the parametric equation and the Cartesian equation of the plane parallel to the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and passing through the point $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

Solution: ► *The desired parametric equation is*

$$\begin{pmatrix} x \\ y + 1 \\ z - 2 \end{pmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

This gives

$$s = z - 2, \qquad t = y + 1 - s = y + 1 - z + 2 = y - z + 3$$

and

$$x = s + t = z - 2 + y - z + 3 = y + 1.$$

Hence the Cartesian equation is $x - y = 1$. ◄

72 Definition If $\vec{n}$ is perpendicular to plane Π_1 and $\vec{n}'$ is perpendicular to plane Π_2, the *angle between the two planes* is the angle between the two vectors $\vec{n}$ and $\vec{n}'$.

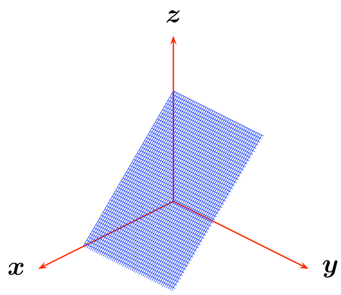

Figure 1.66: The plane $z = 1 - x$.

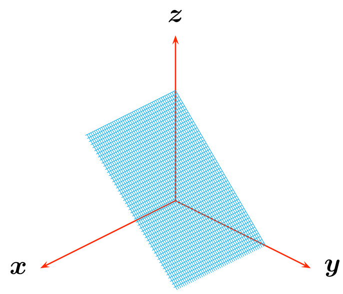

Figure 1.67: The plane $z = 1 - y$.

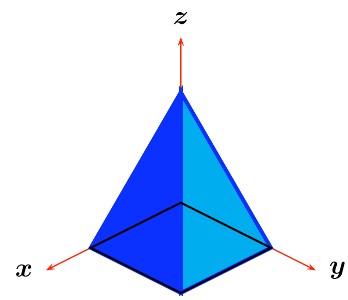

Figure 1.68: Solid bounded by the planes $z = 1 - x$ and $z = 1 - y$ in the first octant.

73 Example

1. Draw the intersection of the plane $z = 1 - x$ with the first octant.
2. Draw the intersection of the plane $z = 1 - y$ with the first octant.
3. Find the angle between the planes $z = 1 - x$ and $z = 1 - y$.
4. Draw the solid $\mathscr{S}$ which results from the intersection of the planes $z = 1 - x$ and $z = 1 - y$ with the first octant.
5. Find the volume of the solid $\mathscr{S}$.

Solution: ▶

1. *This appears in figure 1.66.*
2. *This appears in figure 1.67.*
3. *The vector* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ *is normal to the plane* $x + z = 1$, *and the vector* $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ *is normal to the plane* $y + z = 1$. *If* θ *is the angle between these two vectors, then*

$$\cos\theta = \frac{1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1}{\sqrt{1^2 + 1^2} \cdot \sqrt{1^2 + 1^2}} \implies \cos\theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}.$$

4. *This appears in figure 1.68.*
5. *The resulting solid is a pyramid with square base of area* $A = 1 \cdot 1 = 1$. *Recall that the volume of a pyramid is given by the formula* $V = \dfrac{Ah}{3}$, *where* A *is area of the base of the pyramid and* h *is its height. Now, the height of this pyramid is clearly* 1, *and hence the volume required is* $\dfrac{1}{3}$.

◀

Homework

Problem 1.7.1 Vectors $\vec{a}$, $\vec{b}$ satisfy $\left|\left|\vec{a}\right|\right| = 13$, $\left|\left|\vec{b}\right|\right| = 19$, $\left|\left|\vec{a} + \vec{b}\right|\right| = 24$. Find $\left|\left|\vec{a} - \vec{b}\right|\right|$.

Problem 1.7.2 Find the equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in the direction of $\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$.

Problem 1.7.3 Find the equation of plane containing the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and perpendicular to the line $x = 1 + t, y = -2t, z = 1 - t$.

Problem 1.7.4 Find the equation of plane containing the point $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and containing the line $x = 2y = 3z$.

Problem 1.7.5 (Putnam Exam 1984) Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a > 0, b > 0, c > 0$. Let B be the set of all points which are at distance at most 1 from some point of A (in particular, $A \subseteq B$). Express the volume of B as a polynomial in a, b, c.

Problem 1.7.6 It is known that $\left|\left|\vec{a}\right|\right| = 3$, $\left|\left|\vec{b}\right|\right| = 4$, $\left|\left|\vec{c}\right|\right| = 5$ and that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Find

$$\vec{a} \bullet \vec{b} + \vec{b} \bullet \vec{c} + \vec{c} \bullet \vec{a}.$$

Problem 1.7.7 Find the equation of the line perpendicular to the plane $ax + a^2y + a^3z = 0, \quad a \neq 0$ and passing through the point $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Problem 1.7.8 Find the equation of the plane perpendicular to the line $ax = by = cz, \quad abc \neq 0$ and passing through the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$.

Problem 1.7.9 Find the (shortest) distance from the point $(1, 2, 3)$ to the plane $x - y + z = 1$.

Problem 1.7.10 Determine whether the lines

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

intersect. Find the angle between them.

Problem 1.7.11 Let a, b, c be arbitrary real numbers. Prove that

$$(a^2 + b^2 + c^2)^2 \leq 3(a^4 + b^4 + c^4).$$

Problem 1.7.12 Let $a > 0, b > 0, c > 0$ be the lengths of the sides of $\triangle ABC$. (Vertex A is opposite to the side measuring a, etc.) Recall that by Heron's Formula, the area of this triangle is $S(a, b, c) = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$ is the semiperimeter of the triangle. Prove that $f(a, b, c) = \dfrac{S(a, b, c)}{a^2 + b^2 + c^2}$ is maximised when $\triangle ABC$ is equilateral, and find this maximum.

Problem 1.7.13 Let x, y, z be strictly positive numbers. Prove that

$$\frac{\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}}{\sqrt{x+y+z}} \leq \sqrt{6}.$$

Problem 1.7.14 Let x, y, z be strictly positive numbers. Prove that

$$x + y + z \leq 2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right).$$

Problem 1.7.15 Find the Cartesian equation of the plane passing through $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Draw this plane and its intersection with the first octant. Find the volume of the tetrahedron with vertices at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Problem 1.7.16 Prove that there do not exist three unit vectors in $\mathbb{R}^3$ such that the angle between any two of them be $> \dfrac{2\pi}{3}$.

Problem 1.7.17 Let $(\vec{r} - \vec{a}) \bullet \vec{n} = 0$ be a plane passing through the point $\mathbf{a}$ and perpendicular to vector $\vec{n}$. If $\mathbf{b}$ is not a point on the plane, then the distance from $\mathbf{b}$ to the plane is

$$\frac{\left|(\vec{a} - \vec{b}) \bullet \vec{n}\right|}{\left|\left|\vec{n}\right|\right|}.$$

Problem 1.7.18 (Putnam Exam 1980) Let S be the solid in three-dimensional space consisting of all points $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying the following system of six conditions:

$$x \geq 0, \quad y \geq 0, \quad z \geq 0,$$
$$x + y + z \leq 11,$$
$$2x + 4y + 3z \leq 36,$$
$$2x + 3z \leq 24.$$

Determine the number of vertices and the number of edges of S.

Problem 1.7.19 Given a polyhedron with n faces, consider n vectors, each normal to a face of the polyhedron, and length equal to the area of the face. Prove that the sum of these vectors is $\overrightarrow{0}$.

1.8 Cross Product

We now define the standard cross product in $\mathbb{R}^3$ as a product satisfying the following properties.

74 Definition Let $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$ be vectors in $\mathbb{R}^3$, and let $\alpha \in \mathbb{R}$ be a scalar. The cross product $\times$ is a closed binary operation satisfying

❶ **Anti-commutativity:** $\overrightarrow{x} \times \overrightarrow{y} = -(\overrightarrow{y} \times \overrightarrow{x})$

❷ **Bilinearity:**
$$(\overrightarrow{x} + \overrightarrow{z}) \times \overrightarrow{y} = \overrightarrow{x} \times \overrightarrow{y} + \overrightarrow{z} \times \overrightarrow{y} \quad \text{and} \quad \overrightarrow{x} \times (\overrightarrow{z} + \overrightarrow{y}) = \overrightarrow{x} \times \overrightarrow{z} + \overrightarrow{x} \times \overrightarrow{y}$$

❸ **Scalar homogeneity:** $(\alpha\overrightarrow{x}) \times \overrightarrow{y} = \overrightarrow{x} \times (\alpha\overrightarrow{y}) = \alpha(\overrightarrow{x} \times \overrightarrow{y})$

❹ $\overrightarrow{x} \times \overrightarrow{x} = \overrightarrow{0}$

❺ **Right-hand Rule:**
$$\overrightarrow{i} \times \overrightarrow{j} = \overrightarrow{k}, \quad \overrightarrow{j} \times \overrightarrow{k} = \overrightarrow{i}, \quad \overrightarrow{k} \times \overrightarrow{i} = \overrightarrow{j}.$$

It follows that the cross product is an operation that, given two non-parallel vectors on a plane, allows us to "get out" of that plane.

75 Example Find
$$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Solution: ▸ *We have*
$$\begin{aligned}(\overrightarrow{i} - 3\overrightarrow{k}) \times (\overrightarrow{j} + 2\overrightarrow{k}) &= \overrightarrow{i} \times \overrightarrow{j} + 2\overrightarrow{i} \times \overrightarrow{k} - 3\overrightarrow{k} \times \overrightarrow{j} - 6\overrightarrow{k} \times \overrightarrow{k} \\ &= \overrightarrow{k} - 2\overrightarrow{j} + 3\overrightarrow{i} + 6\overrightarrow{0} \\ &= 3\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}.\end{aligned}$$

Hence
$$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

◂

☞ *The cross product of vectors in* $\mathbb{R}^3$ *is not associative, since*
$$\overrightarrow{i} \times (\overrightarrow{i} \times \overrightarrow{j}) = \overrightarrow{i} \times \overrightarrow{k} = -\overrightarrow{j}$$
but
$$(\overrightarrow{i} \times \overrightarrow{i}) \times \overrightarrow{j} = \overrightarrow{0} \times \overrightarrow{j} = \overrightarrow{0}.$$

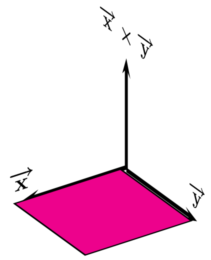

Figure 1.69: Theorem 79.

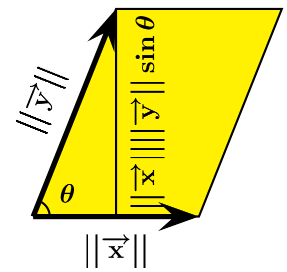

Figure 1.70: Area of a parallelogram

Operating as in example 75 we obtain

76 Theorem Let $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\overrightarrow{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Then

$$\overrightarrow{x} \times \overrightarrow{y} = (x_2y_3 - x_3y_2)\overrightarrow{i} + (x_3y_1 - x_1y_3)\overrightarrow{j} + (x_1y_2 - x_2y_1)\overrightarrow{k}.$$

Proof: *Since* $\overrightarrow{i} \times \overrightarrow{i} = \overrightarrow{j} \times \overrightarrow{j} = \overrightarrow{k} \times \overrightarrow{k} = \overrightarrow{0}$, *we only worry about the mixed products, obtaining,*

$$\begin{aligned} \overrightarrow{x} \times \overrightarrow{y} &= (x_1\overrightarrow{i} + x_2\overrightarrow{j} + x_3\overrightarrow{k}) \times (y_1\overrightarrow{i} + y_2\overrightarrow{j} + y_3\overrightarrow{k}) \\ &= x_1y_2\overrightarrow{i} \times \overrightarrow{j} + x_1y_3\overrightarrow{i} \times \overrightarrow{k} + x_2y_1\overrightarrow{j} \times \overrightarrow{i} + x_2y_3\overrightarrow{j} \times \overrightarrow{k} \\ &\quad + x_3y_1\overrightarrow{k} \times \overrightarrow{i} + x_3y_2\overrightarrow{k} \times \overrightarrow{j} \\ &= (x_1y_2 - y_1x_2)\overrightarrow{i} \times \overrightarrow{j} + (x_2y_3 - x_3y_2)\overrightarrow{j} \times \overrightarrow{k} + (x_3y_1 - x_1y_3)\overrightarrow{k} \times \overrightarrow{i} \\ &= (x_1y_2 - y_1x_2)\overrightarrow{k} + (x_2y_3 - x_3y_2)\overrightarrow{i} + (x_3y_1 - x_1y_3)\overrightarrow{j}, \end{aligned}$$

proving the theorem. ❑

Using the cross product, we may obtain a third vector simultaneously perpendicular to two other vectors in space.

77 Theorem $\overrightarrow{x} \perp (\overrightarrow{x} \times \overrightarrow{y})$ and $\overrightarrow{y} \perp (\overrightarrow{x} \times \overrightarrow{y})$, that is, the cross product of two vectors is simultaneously perpendicular to both original vectors.

Proof: *We will only check the first assertion, the second verification is analogous.*

$$\begin{aligned} \overrightarrow{x} \bullet (\overrightarrow{x} \times \overrightarrow{y}) &= (x_1\overrightarrow{i} + x_2\overrightarrow{j} + x_3\overrightarrow{k}) \bullet ((x_2y_3 - x_3y_2)\overrightarrow{i} \\ &\quad + (x_3y_1 - x_1y_3)\overrightarrow{j} + (x_1y_2 - x_2y_1)\overrightarrow{k}) \\ &= x_1x_2y_3 - x_1x_3y_2 + x_2x_3y_1 - x_2x_1y_3 + x_3x_1y_2 - x_3x_2y_1 \\ &= 0, \end{aligned}$$

completing the proof. ❑

Although the cross product is not associative, we have, however, the following theorem.

78 Theorem

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a} \bullet \overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a} \bullet \overrightarrow{b})\overrightarrow{c}.$$

Proof:

$$\begin{aligned}\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c}) &= (a_1\overrightarrow{i}+a_2\overrightarrow{j}+a_3\overrightarrow{k})\times((b_2c_3-b_3c_2)\overrightarrow{i}+\\ &\qquad +(b_3c_1-b_1c_3)\overrightarrow{j}+(b_1c_2-b_2c_1)\overrightarrow{k})\\ &= a_1(b_3c_1-b_1c_3)\overrightarrow{k}-a_1(b_1c_2-b_2c_1)\overrightarrow{j}-a_2(b_2c_3-b_3c_2)\overrightarrow{k}\\ &\qquad +a_2(b_1c_2-b_2c_1)\overrightarrow{i}+a_3(b_2c_3-b_3c_2)\overrightarrow{j}-a_3(b_3c_1-b_1c_3)\overrightarrow{i}\\ &= (a_1c_1+a_2c_2+a_3c_3)(b_1\overrightarrow{i}+b_2\overrightarrow{j}+b_3\overrightarrow{i})+\\ &\qquad (-a_1b_1-a_2b_2-a_3b_3)(c_1\overrightarrow{i}+c_2\overrightarrow{j}+c_3\overrightarrow{i})\\ &= (\overrightarrow{a}\bullet\overrightarrow{c})\overrightarrow{b}-(\overrightarrow{a}\bullet\overrightarrow{b})\overrightarrow{c},\end{aligned}$$

completing the proof. ❑

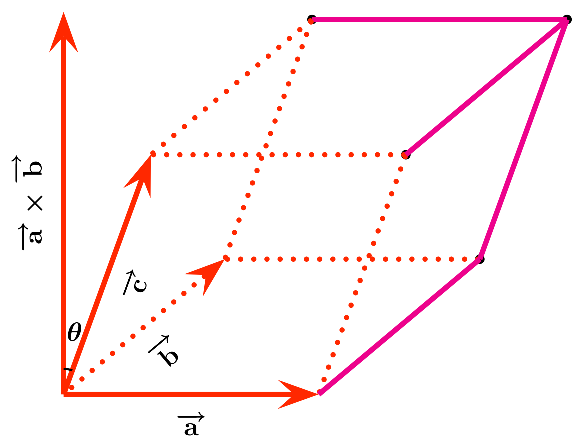

Figure 1.71: Theorem 82.

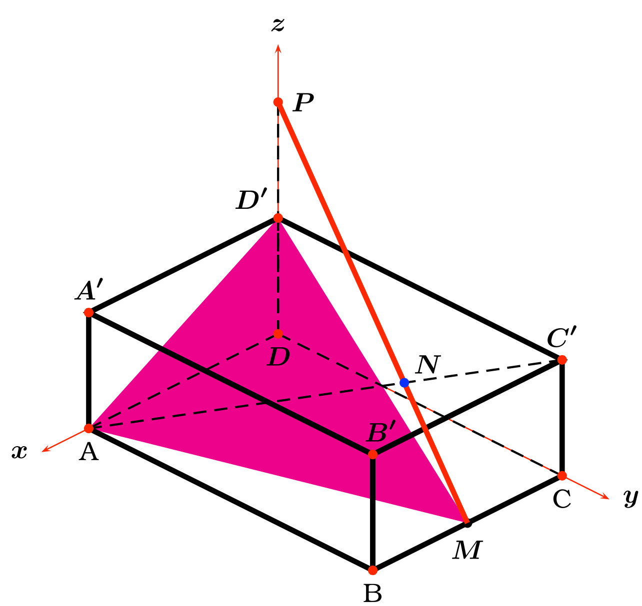

Figure 1.72: Example 83.

79 Theorem Let $(\widehat{\overrightarrow{x},\overrightarrow{y}})\in[0;\pi]$ be the convex angle between two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$. Then

$$||\overrightarrow{x}\times\overrightarrow{y}|| = ||\overrightarrow{x}||||\overrightarrow{y}||\sin(\widehat{\overrightarrow{x},\overrightarrow{y}}).$$

Proof: *We have*

$$\begin{aligned}||\overrightarrow{x}\times\overrightarrow{y}||^2 &= (x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2+(x_1y_2-x_2y_1)^2\\ &= y^2y_3^2-2x_2y_3x_3y_2+z^2y_2^2+z^2y_1^2-2x_3y_1x_1y_3+\\ &\qquad +x^2y_3^2+x^2y_2^2-2x_1y_2x_2y_1+y^2y_1^2\\ &= (x^2+y^2+z^2)(y_1^2+y_2^2+y_3^2)-(x_1y_1+x_2y_2+x_3y_3)^2\\ &= ||\overrightarrow{x}||^2||\overrightarrow{y}||^2-(\overrightarrow{x}\bullet\overrightarrow{y})^2\\ &= ||\overrightarrow{x}||^2||\overrightarrow{y}||^2-||\overrightarrow{x}||^2||\overrightarrow{y}||^2\cos^2(\widehat{\overrightarrow{x},\overrightarrow{y}})\\ &= ||\overrightarrow{x}||^2||\overrightarrow{y}||^2\sin^2(\widehat{\overrightarrow{x},\overrightarrow{y}}),\end{aligned}$$

whence the theorem follows. ❑

Theorem 79 has the following geometric significance: $\left|\left|\overrightarrow{x} \times \overrightarrow{y}\right|\right|$ is the area of the parallelogram formed when the tails of the vectors are joined. See figure 1.70.

The following corollaries easily follow from Theorem 79.

80 Corollary Two non-zero vectors $\overrightarrow{x}, \overrightarrow{y}$ satisfy $\overrightarrow{x} \times \overrightarrow{y} = \overrightarrow{0}$ if and only if they are parallel.

81 Corollary (Lagrange's Identity)

$$||\overrightarrow{x} \times \overrightarrow{y}||^2 = ||x||^2||y||^2 - (\overrightarrow{x}\bullet\overrightarrow{y})^2.$$

The following result mixes the dot and the cross product.

82 Theorem Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, be linearly independent vectors in $\mathbb{R}^3$. The signed volume of the parallelepiped spanned by them is $(\overrightarrow{a} \times \overrightarrow{b}) \bullet \overrightarrow{c}$.

Proof: *See figure 1.71. The area of the base of the parallelepiped is the area of the parallelogram determined by the vectors* $\overrightarrow{a}$ *and* $\overrightarrow{b}$*, which has area* $\left|\left|\overrightarrow{a} \times \overrightarrow{b}\right|\right|$*. The altitude of the parallelepiped is* $\left|\left|\overrightarrow{c}\right|\right| \cos\theta$ *where* θ *is the angle between* $\overrightarrow{c}$ *and* $\overrightarrow{a} \times \overrightarrow{b}$*. The volume of the parallelepiped is thus*

$$\left|\left|\overrightarrow{a} \times \overrightarrow{b}\right|\right|\left|\left|\overrightarrow{c}\right|\right| \cos\theta = (\overrightarrow{a} \times \overrightarrow{b}) \bullet \overrightarrow{c},$$

proving the theorem. ❑

☞ *Since we may have used any of the faces of the parallelepiped, it follows that*

$$(\overrightarrow{a} \times \overrightarrow{b}) \bullet \overrightarrow{c} = (\overrightarrow{b} \times \overrightarrow{c}) \bullet \overrightarrow{a} = (\overrightarrow{c} \times \overrightarrow{a}) \bullet \overrightarrow{b}.$$

In particular, it is possible to "exchange" the cross and dot products:

$$\overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a} \times \overrightarrow{b}) \bullet \overrightarrow{c}$$

83 Example Consider the rectangular parallelepiped $ABCDD'C'B'A'$ (figure 1.72) with vertices $A(2,0,0)$, $B(2,3,0)$, $C(0,3,0)$, $D(0,0,0)$, $D'(0,0,1)$, $C'(0,3,1)$, $B'(2,3,1)$, $A'(2,0,1)$. Let M be the midpoint of the line segment joining the vertices B and C.

1. Find the Cartesian equation of the plane containing the points A, D', and M.
2. Find the area of $\triangle AD'M$.
3. Find the parametric equation of the line $\overleftrightarrow{AC'}$.
4. Suppose that a line through M is drawn cutting the line segment $[AC']$ in N and the line $\overleftrightarrow{DD'}$ in P. Find the parametric equation of $\overleftrightarrow{MP}$.

Solution: ►

1. *Form the following vectors and find their cross product:*

$$\overrightarrow{AD'} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad \overrightarrow{AM} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \implies \overrightarrow{AD'} \times \overrightarrow{AM} = \begin{bmatrix} -3 \\ -1 \\ -6 \end{bmatrix}.$$

The equation of the plane is thus

$$\begin{bmatrix} x-2 \\ y-0 \\ z-0 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ -1 \\ -6 \end{bmatrix} = 0 \implies 3(x-2) + 1(y) + 6z = 0 \implies 3x + y + 6z = 6.$$

2. *The area of the triangle is*

$$\frac{\left\|\overrightarrow{\mathbf{AD'}} \times \overrightarrow{\mathbf{AM}}\right\|}{2} = \frac{1}{2}\sqrt{3^2 + 1^2 + 6^2} = \frac{\sqrt{46}}{2}.$$

3. *We have* $\overrightarrow{\mathbf{AC'}} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$*, and hence the line* $\overleftrightarrow{\mathbf{AC'}}$ *has parametric equation*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \implies x = 2 - 2t,\ y = 3t,\ z = t.$$

4. *Since* P *is on the* z*-axis,* $P = \begin{pmatrix} 0 \\ 0 \\ z' \end{pmatrix}$ *for some real number* $z' > 0$*. The parametric equation of the line* $\overleftrightarrow{\mathbf{MP}}$ *is thus*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s \begin{bmatrix} -1 \\ -3 \\ z' \end{bmatrix} \implies x = 1 - s \qquad y = 3 - 3s, \qquad z = sz'.$$

Since N *is on both* $\overleftrightarrow{\mathbf{MP}}$ *and* $\overleftrightarrow{\mathbf{AC'}}$ *we must have*

$$2 - 2t = 1 - s,\ 3t = 3 - 3s,\ t = sz'.$$

Solving the first two equations gives $s = \frac{1}{3}, t = \frac{2}{3}$*. Putting this into the third equation we deduce* $z' = 2$*. Thus* $P = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ *and the desired equation is*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \implies x = 1 - s, \quad y = 3 - 3s, \qquad z = 2s.$$

◀

Homework

Problem 1.8.1 Prove that

$$(\overrightarrow{a} - \overrightarrow{b}) \times (\overrightarrow{a} + \overrightarrow{b}) = 2\overrightarrow{a} \times \overrightarrow{b}.$$

Problem 1.8.2 Prove that $\overrightarrow{x} \times \overrightarrow{x} = \overrightarrow{0}$ follows from the anti-commutativity of the cross product.

Problem 1.8.3 If $\overrightarrow{b} - \overrightarrow{a}$ and $\overrightarrow{c} - \overrightarrow{a}$ are parallel and it is known that $\overrightarrow{c} \times \overrightarrow{a} = \overrightarrow{i} - \overrightarrow{j}$ and $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{j} + \overrightarrow{k}$, find $\overrightarrow{b} \times \overrightarrow{c}$.

Problem 1.8.4 Redo example 71, that is, find the Cartesian equation of the plane parallel to the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and passing through the point $(0, -1, 2)$, by finding a normal to the plane.

Problem 1.8.5 Find the equation of the plane passing through the points $(a, 0, a)$, $(-a, 1, 0)$, and $(0, 1, 2a)$ in $\mathbb{R}^3$.

Problem 1.8.6 Let $a \in \mathbb{R}$. Find a vector of unit length simultaneously perpendicular to $\overrightarrow{v} = \begin{bmatrix} 0 \\ -a \\ a \end{bmatrix}$ and $\overrightarrow{w} = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}$.

Problem 1.8.7 (Jacobi's Identity) Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ be vectors in $\mathbb{R}^3$. Prove that

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) + \overrightarrow{b} \times (\overrightarrow{c} \times \overrightarrow{a}) + \overrightarrow{c} \times (\overrightarrow{a} \times \overrightarrow{b}) = \overrightarrow{0}.$$

Problem 1.8.8 Let $\overrightarrow{x} \in \mathbb{R}^3, ||\mathbf{x}|| = 1$. Find

$$||\overrightarrow{x} \times \overrightarrow{i}||^2 + ||\overrightarrow{x} \times \overrightarrow{j}||^2 + ||\overrightarrow{x} \times \overrightarrow{k}||^2.$$

Problem 1.8.9 The vectors $\overrightarrow{a}$, $\overrightarrow{b}$ are constant vectors. Solve the equation

$$\overrightarrow{a} \times (\overrightarrow{x} \times \overrightarrow{b}) = \overrightarrow{b} \times (\overrightarrow{x} \times \overrightarrow{a}).$$

Problem 1.8.10 If $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$, prove that

$$\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{b} \times \overrightarrow{c} = \overrightarrow{c} \times \overrightarrow{a}.$$

Problem 1.8.11 Assume $\overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c}) \neq 0$ and that

$$\overrightarrow{x} = \alpha \overrightarrow{a} + \beta \overrightarrow{b} + \gamma \overrightarrow{c}.$$

Find α, β, and γ in terms of $\overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c})$.

Problem 1.8.12 The vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are constant vectors. Solve the system of equations

$$2\overrightarrow{x} + \overrightarrow{y} \times \overrightarrow{a} = \overrightarrow{b}, \quad 3\overrightarrow{y} + \overrightarrow{x} \times \overrightarrow{a} = \overrightarrow{c},$$

Problem 1.8.13 Let $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}$ be vectors in $\mathbb{R}^3$. Prove the following vector identity,

$$(\overrightarrow{a} \times \overrightarrow{b}) \bullet (\overrightarrow{c} \times \overrightarrow{d}) = (\overrightarrow{a} \bullet \overrightarrow{c})(\overrightarrow{b} \bullet \overrightarrow{d}) - (\overrightarrow{a} \bullet \overrightarrow{d})(\overrightarrow{b} \bullet \overrightarrow{c}).$$

Problem 1.8.14 Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$, be vectors in $\mathbb{R}^3$. Prove that

$$\begin{aligned} &(\overrightarrow{b} \times \overrightarrow{c}) \bullet (\overrightarrow{a} \times \overrightarrow{d}) \\ &\qquad + (\overrightarrow{c} \times \overrightarrow{a}) \bullet (\overrightarrow{b} \times \overrightarrow{d}) \\ &\qquad + (\overrightarrow{a} \times \overrightarrow{b}) \bullet (\overrightarrow{c} \times \overrightarrow{d}) \\ &\qquad = 0. \end{aligned}$$

Problem 1.8.15 Consider the plane Π passing through the points $A(6, 0, 0)$, $B(0, 4, 0)$ and $C(0, 0, 3)$, as shewn in figure 1.73 below. The plane Π intersects a $3 \times 3 \times 3$ cube, one of whose vertices is at the origin and that has three of its edges on the coordinate axes, as in the figure. This intersection forms a pentagon $CPQRS$.

1. Find $\overrightarrow{\mathbf{CA}} \times \overrightarrow{\mathbf{CB}}$.
2. Find $\left|\left|\overrightarrow{\mathbf{CA}} \times \overrightarrow{\mathbf{CB}}\right|\right|$.
3. Find the parametric equation of the line L_{CA} joining C and A, with a parameter $t \in \mathbb{R}$.
4. Find the parametric equation of the line L_{DE} joining D and E, with a parameter $s \in \mathbb{R}$.
5. Find the intersection point between the lines L_{CA} and L_{DE}.
6. Find the equation of the plane Π.
7. Find the area of $\triangle ABC$.
8. Find the coordinates of the points P, Q, R, and S.
9. Find the area of the pentagon $CPQRS$.

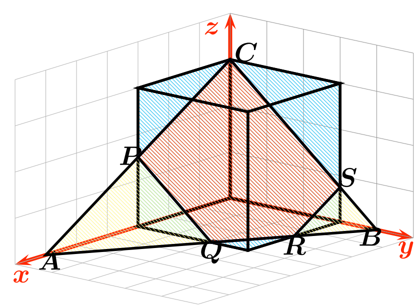

Figure 1.73: Problem 1.8.15..

1.9 Matrices in three dimensions

We will briefly introduce 3×3 matrices. Most of the material will flow like that for 2×2 matrices.

84 Definition A *linear transformation* $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a function such that

$$T(\mathrm{a} + \mathrm{b}) = T(\mathrm{a}) + T(\mathrm{b}), \qquad T(\lambda \mathrm{a}) = \lambda T(\mathrm{a}),$$

for all points a, b in $\mathbb{R}^3$ and all scalars λ. Such a linear transformation has a 3×3 matrix representation whose columns are the vectors $T(\mathrm{i})$, $T(\mathrm{j})$, and $T(\mathrm{k})$.

85 Example Consider $L : \mathbb{R}^3 \to \mathbb{R}^3$, with

$$L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 - x_3 \\ x_1 + x_2 + x_3 \\ x_3 \end{pmatrix}.$$

❶ Prove that L is a linear transformation.

❷ Find the matrix corresponding to L under the standard basis.

Solution: ►

❶ *Let $\alpha \in \mathbb{R}$ and let $\mathbf{u}, \mathbf{v}$ be points in $\mathbb{R}^3$. Then*

$$\begin{aligned} L(\mathbf{u}+\mathbf{v}) &= L\left(\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}\right) \\ &= \begin{pmatrix} (u_1+v_1)-(u_2+v_2)-(u_3+v_3) \\ (u_1+v_1)+(u_2+v_2)+(u_3+v_3) \\ u_3+v_3 \end{pmatrix} \\ &= \begin{pmatrix} u_1-u_2-u_3 \\ u_1+u_2+u_3 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1-v_2-v_3 \\ v_1+v_2+v_3 \\ v_3 \end{pmatrix} \\ &= L\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right) + L\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) \\ &= L(\mathbf{u}) + L(\mathbf{v}), \end{aligned}$$

and also

$$\begin{aligned} L(\alpha\mathbf{u}) &= L\left(\begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{pmatrix}\right) \\ &= \begin{pmatrix} \alpha(u_1)-\alpha(u_2)-\alpha(u_3) \\ \alpha(u_1)+\alpha(u_2)+\alpha(u_3) \\ \alpha u_3 \end{pmatrix} \\ &= \alpha\begin{pmatrix} u_1-u_2-u_3 \\ u_1+u_2+u_3 \\ u_3 \end{pmatrix} \\ &= \alpha L\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right) \\ &= \alpha L(\mathbf{u}), \end{aligned}$$

proving that L is a linear transformation.

❷ *We have* $L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, *and* $L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, *whence the* *desired matrix is*

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

◀

Addition, scalar multiplication, and matrix multiplication are defined for 3×3 matrices in a manner analogous to those operations for 2×2 matrices.

86 Definition Let A, B be 3×3 matrices. Then we define

$$A + B = [a_{ij} + b_{ij}], \qquad \alpha A = [\alpha a_{ij}], \qquad AB = \left[\sum_{k=1}^{3} a_{ik}b_{kj}\right].$$

87 Example If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} a & b & c \\ a & b & 0 \\ a & 0 & 0 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1+a & 2+b & 3+c \\ 4+a & 5+b & 0 \\ 6+a & 0 & 0 \end{bmatrix}, \qquad 3A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 0 \\ 18 & 0 & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} a+4b+6c & 2a+5b & 3a \\ a+4b & 2a+5b & 3a \\ a & 2a & 3a \end{bmatrix}, \qquad AB = \begin{bmatrix} 6a & 3b & c \\ 9a & 9b & 4c \\ 6a & 6b & 6c \end{bmatrix}.$$

88 Definition A *scaling matrix* is one of the form

$$S_{a,b,c} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

where $a > 0$, $b > 0$, $c > 0$.

It is an easy exercise to prove that the product of two scaling matrices commutes.

89 Definition A *rotation matrix* about the z-axis by an angle θ in the counterclockwise sense is

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A *rotation matrix* about the y-axis by an angle θ in the counterclockwise sense is

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

A *rotation matrix* about the x-axis by an angle θ in the counterclockwise sense is

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Easy to find counterexamples should convince the reader that the product of two rotations in space does not necessarily commute.

90 Definition A *reflexion matrix* about the x-axis is

$$R_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A *reflexion matrix* about the y-axis is

$$R_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A *reflexion matrix* about the z-axis is

$$R_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Homework

Problem 1.9.1 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find A^2, A^3 and A^4. Conjecture and, prove by induction, a general formula for A^n.

Problem 1.9.2 Let $A \in \mathrm{M}_{3\times 3}(\mathbb{R})$ be given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Demonstrate, using induction, that $A^n = 3^{n-1}A$ for $n \in \mathbb{N}, n \geq 1$.

Problem 1.9.3 Consider the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Describe A^2 and A^3 in terms of n.

Problem 1.9.4 Let x be a real number, and put

$$m(x) = \begin{bmatrix} 1 & 0 & x \\ -x & 1 & -\frac{x^2}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

If a, b are real numbers, prove that

1. $m(a)m(b) = m(a+b)$.
2. $m(a)m(-a) = \mathbf{I}_3$, the 3×3 identity matrix.

1.10 Determinants in three dimensions

We now define the notion of *determinant* of a 3×3 matrix. Consider now the vectors $\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $\overrightarrow{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, in $\mathbb{R}^3$, and the 3×3 matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. Since thanks to Theorem 82, the volume of the parallelepiped spanned by these vectors is $\overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c})$, we *define* the determinant of A, $\det A$, to be

$$D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c}). \tag{1.15}$$

We now establish that the properties of the determinant of a 3×3 as defined above are analogous to those of the determinant of 2×2 matrix defined in the preceding chapter.

91 Theorem The determinant of a 3×3 matrix A as defined by (1.15) satisfies the following properties:

1. D is linear in each of its arguments.

2. If the parallelepiped is flat then the volume is 0, that is, if $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, are linearly dependent, then $D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = 0$.

3. $D(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}) = 1$, and accords with the right-hand rule.

Proof:

1. *If* $D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = \overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c})$, *linearity of the first component follows by the distributive law for the dot product:*

$$\begin{aligned} D(\overrightarrow{a} + \overrightarrow{a}', \overrightarrow{b}, \overrightarrow{c}) &= (\overrightarrow{a} + \overrightarrow{a}') \bullet (\overrightarrow{b} \times \overrightarrow{c}) \\ &= \overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c}) + \overrightarrow{a}' \bullet (\overrightarrow{b} \times \overrightarrow{c}) \\ &= D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) + D(\overrightarrow{a}', \overrightarrow{b}, \overrightarrow{c}), \end{aligned}$$

and if $\lambda \in \mathbb{R}$,

$$D(\lambda \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = (\lambda \overrightarrow{a}) \bullet (\overrightarrow{b} \times \overrightarrow{c}) = \lambda((\overrightarrow{a}) \bullet (\overrightarrow{b} \times \overrightarrow{c})) = \lambda D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}).$$

The linearity on the second and third component can be established by using the distributive law of the cross product. For example, for the second component we have,

$$\begin{aligned} D(\overrightarrow{a}, \overrightarrow{b} + \overrightarrow{b}', \overrightarrow{c}) &= \overrightarrow{a} \bullet ((\overrightarrow{b} + \overrightarrow{b}') \times \overrightarrow{c}) \\ &= \overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c} + \overrightarrow{b'} \times \overrightarrow{c}) \\ &= \overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c}) + \overrightarrow{a} \bullet (\overrightarrow{b'} \times \overrightarrow{c}) \\ &= D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) + D(\overrightarrow{a}, \overrightarrow{b'}, \overrightarrow{c}), \end{aligned}$$

and if $\lambda \in \mathbb{R}$,

$$D(\overrightarrow{a}, \lambda \overrightarrow{b}, \overrightarrow{c}) = \overrightarrow{a} \bullet ((\lambda \overrightarrow{b}) \times \overrightarrow{c}) = \lambda(\overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c})) = \lambda D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}).$$

2. *If* $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, *are linearly dependent, then they lie on the same plane and the parallelepiped spanned by them is flat, hence,* $D(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = 0$.

3. *Since* $\overrightarrow{j} \times \overrightarrow{k} = \overrightarrow{i}$, *and* $\overrightarrow{i} \bullet \overrightarrow{i} = 1$,

$$D(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}) = \overrightarrow{i} \bullet (\overrightarrow{j} \times \overrightarrow{k}) = \overrightarrow{i} \bullet \overrightarrow{i} = 1.$$

❑

Observe that

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c}) \quad (1.16)$$

$$= \overrightarrow{a} \bullet \left((b_2c_3 - b_3c_2)\overrightarrow{i} + (b_3c_1 - b_1c_3)\overrightarrow{j} + (b_1c_2 - b_2c_1)\overrightarrow{k}\right) \quad (1.17)$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \quad (1.18)$$

$$= a_1 \det \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}, \quad (1.19)$$

which reduces the computation of 3×3 determinants to 2×2 determinants.

92 Example Find $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution: ► *Using (1.19), we have*

$$\begin{aligned} \det A &= 1 \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 4 \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 7 \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \\ &= 1(45 - 48) - 4(18 - 24) + 7(12 - 15) \\ &= -3 + 24 - 21 \\ &= 0. \end{aligned}$$

◄

Again, we may use the Maple™ packages `linalg`, `LinearAlgebra`, or `Student[VectorCalculus]` to perform many of the vector operations. An example follows with `linalg`.

```
> with(linalg):
> a:=vector([-2,0,1]);
```

$$a := [-2, 0, 1]$$

```
> b:=vector([-1,3,0]);
```

$$b := [-1, 3, 0]$$

```
> crossprod(a,b);
```

$$[-3, -1, -6]$$

```
> dotprod(a,b);
```

$$2$$

```
> angle(a,b);
```

$$\arccos\left(\frac{\sqrt{50}}{25}\right)$$

Homework

Problem 1.10.1 Prove that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b - c)(c - a)(a - b).$$

Problem 1.10.2 Prove that

$$\det \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 3abc - a^3 - b^3 - c^3.$$

1.11 Some Solid Geometry

In this section we examine some examples and prove some theorems of three-dimensional geometry.

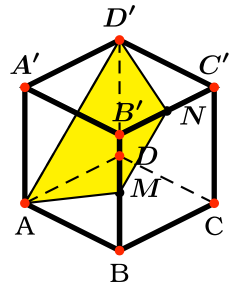

Figure 1.74: Example 93.

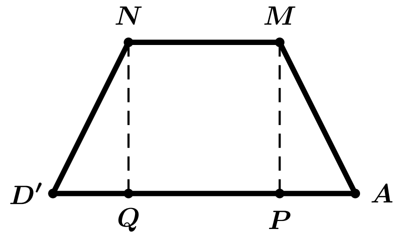

Figure 1.75: Example 93.

93 Example Cube $\mathbf{ABCDD'C'B'A}$ in figure 1.74 has side of length a. M is the midpoint of edge $[\mathbf{BB'}]$ and N is the midpoint of edge $[\mathbf{B'C'}]$. Prove that $\overrightarrow{\mathbf{AD'}} \parallel \overrightarrow{\mathbf{MN}}$ and find the area of the quadrilateral $\mathbf{MND'A}$.

> **Solution:** ► *By the Pythagorean Theorem,* $\left|\left|\overrightarrow{\mathbf{AD'}}\right|\right| = a\sqrt{2}$. *Because they are diagonals that belong to parallel faces of the cube,* $\overrightarrow{\mathbf{AD'}} \parallel \overrightarrow{\mathbf{BC'}}$. *Now,* M *and* N *are the midpoints of the sides* $[B'B]$ *and* $[B'C']$ *of* $\triangle B'C'B$, *and hence* $\overrightarrow{\mathbf{MN}} \parallel \overrightarrow{\mathbf{BC'}}$ *by example 14. The aforementioned example also gives* $\left|\left|\overrightarrow{\mathbf{MN}}\right|\right| = \frac{1}{2}\left|\left|\overrightarrow{\mathbf{AD'}}\right|\right| = \frac{a\sqrt{2}}{2}$. *In consequence,* $\overrightarrow{\mathbf{AD'}} \parallel \overrightarrow{\mathbf{MN}}$. *This means that the four points* $\mathbf{A}, \mathbf{D'}, \mathbf{M}, \mathbf{N}$ *are all on the same plane. Hence* $\mathbf{MND'A}$ *is a trapezoid with bases of length* $a\sqrt{2}$ *and* $\frac{a\sqrt{2}}{2}$ *(figure 1.75). From the figure*
>
> $$\left|\left|\overrightarrow{\mathbf{D'Q}}\right|\right| = \left|\left|\overrightarrow{\mathbf{AP}}\right|\right| = \frac{1}{2}\left(\left|\left|\overrightarrow{\mathbf{AD'}}\right|\right| - \left|\left|\overrightarrow{\mathbf{MN}}\right|\right|\right) = \frac{a\sqrt{2}}{4}.$$
>
> *Also, by the Pythagorean Theorem,*
>
> $$\left|\left|\overrightarrow{\mathbf{D'N}}\right|\right| = \sqrt{\left|\left|\overrightarrow{\mathbf{D'C'}}\right|\right|^2 + \left|\left|\overrightarrow{\mathbf{C'N}}\right|\right|^2} = \sqrt{a^2 + \frac{a^2}{4}} = \frac{a\sqrt{5}}{2}.$$
>
> *The height of this trapezoid is thus*
>
> $$\left|\left|\overrightarrow{\mathbf{NQ}}\right|\right| = \sqrt{\frac{5a^2}{4} - \frac{a^2}{8}} = \frac{3a}{2\sqrt{2}}.$$
>
> *The area of the trapezoid is finally,*
>
> $$\frac{3a}{2\sqrt{2}} \cdot \left(\frac{a\sqrt{2} + \frac{a\sqrt{2}}{2}}{2}\right) = \frac{9a^2}{8}.$$
>
> ◄

Let us prove a three-dimensional version of Thales' Theorem.

94 Theorem (Thales' Theorem) Of two lines are cut by three parallel planes, their corresponding segments are proportional.

Proof: *See figure 1.76. Given the lines* $\overleftrightarrow{\mathrm{AB}}$ *and* $\overleftrightarrow{\mathrm{CD}}$*, we must prove that*

$$\frac{\overline{AE}}{\overline{EB}} = \frac{\overline{CF}}{\overline{FD}}.$$

Draw line $\overleftrightarrow{\mathrm{AD}}$ *cutting plane* $P2$ *in* G*. The plane containing points* A, B*, and* D *intersects plane* $P2$ *in the line* $\overleftrightarrow{\mathrm{EG}}$*. Similarly the plane containing points* A, C*, and* D *intersects plane* $P2$ *in the line* $\overleftrightarrow{\mathrm{GF}}$*. Since* $P2$ *and* $P3$ *are parallel planes,* $\overleftrightarrow{\mathrm{EG}} \parallel \overleftrightarrow{\mathrm{BD}}$*, and so by Thales' Theorem on the plane (theorem 30),*

$$\frac{\overline{AE}}{\overline{EB}} = \frac{\overline{AG}}{\overline{GD}}.$$

Similarly, since $P1$ *and* $P2$ *are parallel,* $\overleftrightarrow{\mathrm{AC}} \parallel \overleftrightarrow{\mathrm{GF}}$ *and*

$$\frac{\overline{CF}}{\overline{FD}} = \frac{\overline{AG}}{\overline{GD}}.$$

It follows that

$$\frac{\overline{AE}}{\overline{EB}} = \frac{\overline{CF}}{\overline{FD}},$$

as needed to be shewn. ❑

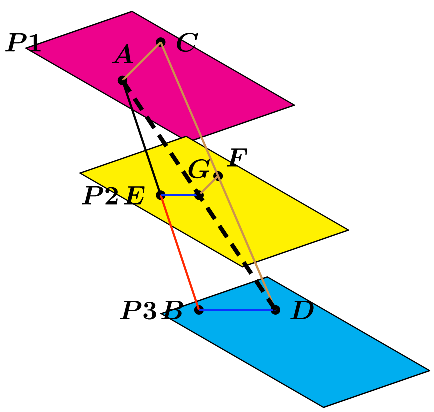

Figure 1.76: Thales' Theorem in 3D.

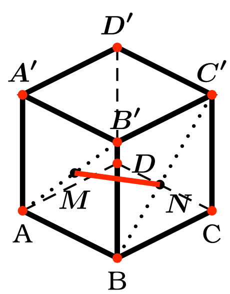

Figure 1.77: Example 95.

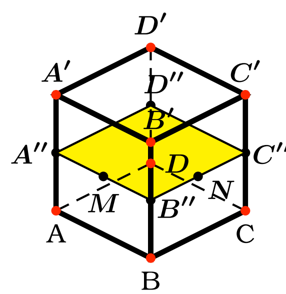

Figure 1.78: Example 95.

95 Example In cube $\mathrm{ABCDD'C'B'A'}$ of edge of length a, the points M and N are located on diagonals $[\mathrm{AB'}]$ and $[\mathrm{BC'}]$ such that $\overrightarrow{\mathrm{MN}}$ is parallel to the face ABCD of the cube. If $\left|\left|\overrightarrow{\mathrm{MN}}\right|\right| = \frac{\sqrt{5}}{3}\left|\left|\overrightarrow{\mathrm{AB}}\right|\right|$, find the ratios $\dfrac{\left|\left|\overrightarrow{\mathrm{AM}}\right|\right|}{\left|\left|\overrightarrow{\mathrm{AB'}}\right|\right|}$ and $\dfrac{\left|\left|\overrightarrow{\mathrm{BN}}\right|\right|}{\left|\left|\overrightarrow{\mathrm{BC'}}\right|\right|}$.

Solution: ▶ *There is a unique plane parallel* P *to face* ABCD *and containing* M*. Since* $\overrightarrow{\mathrm{MN}}$ *is parallel to face* ABCD *,* P *also contains* N*. The intersection of* P *with the cube produces a lamina* $\mathrm{A''B''C''D''}$*, as in figure 1.78.*

First notice that $\left\|\overrightarrow{\mathbf{AB'}}\right\| = \left\|\overrightarrow{\mathbf{BC'}}\right\| = a\sqrt{2}$. *Put*

$$\frac{\left\|\overrightarrow{\mathbf{AM}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|} = x \implies \frac{\left\|\overrightarrow{\mathbf{MB'}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|} = \frac{\left\|\overrightarrow{\mathbf{AB'}}\right\| - \left\|\overrightarrow{\mathbf{AM}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|} = 1 - x.$$

Now, as $\triangle B'AB \sim \triangle B'MB''$ *and* $\triangle BC'B' \sim \triangle BNB''$,

$$\frac{\left\|\overrightarrow{\mathbf{MB'}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|} = \frac{\left\|\overrightarrow{\mathbf{B''B'}}\right\|}{\left\|\overrightarrow{\mathbf{BB'}}\right\|}, \qquad \frac{\left\|\overrightarrow{\mathbf{MB'}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|} = \frac{\left\|\overrightarrow{\mathbf{MB''}}\right\|}{\left\|\overrightarrow{\mathbf{AB}}\right\|} \implies \left\|\overrightarrow{\mathbf{MB''}}\right\| = (1-x)a,$$

$$\frac{\left\|\overrightarrow{\mathbf{BB''}}\right\|}{\left\|\overrightarrow{\mathbf{BB'}}\right\|} = \frac{\left\|\overrightarrow{\mathbf{AM}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|}, \qquad \frac{\left\|\overrightarrow{\mathbf{B''N}}\right\|}{\left\|\overrightarrow{\mathbf{B'C'}}\right\|} = \frac{\left\|\overrightarrow{\mathbf{BB''}}\right\|}{\left\|\overrightarrow{\mathbf{BB'}}\right\|} \implies \left\|\overrightarrow{\mathbf{B''N}}\right\| = xa.$$

Since $\left\|\overrightarrow{\mathbf{MN}}\right\| = \frac{\sqrt{5}}{3}a$, *by the Pythagorean Theorem,*

$$\left\|\overrightarrow{\mathbf{MN}}\right\|^2 = \left\|\overrightarrow{\mathbf{MB''}}\right\|^2 + \left\|\overrightarrow{\mathbf{B''N}}\right\|^2 \implies \frac{5}{9}a^2 = (1-x)^2a^2 + x^2a^2 \implies x \in \left\{\frac{1}{3}, \frac{2}{3}\right\}.$$

There are two possible positions for the segment, giving the solutions

$$\frac{\left\|\overrightarrow{\mathbf{AM}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|} = \frac{\left\|\overrightarrow{\mathbf{BN}}\right\|}{\left\|\overrightarrow{\mathbf{BC'}}\right\|} = \frac{1}{3}, \qquad \frac{\left\|\overrightarrow{\mathbf{AM}}\right\|}{\left\|\overrightarrow{\mathbf{AB'}}\right\|} = \frac{\left\|\overrightarrow{\mathbf{BN}}\right\|}{\left\|\overrightarrow{\mathbf{BC'}}\right\|} = \frac{2}{3}.$$

◀

Homework

Problem 1.11.1 In a regular tetrahedron with vertices A, B, C, D and with $\left\|\overrightarrow{\mathbf{AB}}\right\| = a$, points M and N are the midpoints of the edges $[AB]$ and $[CD]$, respectively.

1. Find the length of the segment $[MN]$.
2. Find the angle between the lines $[MN]$ and $[BC]$.
3. Prove that $\overrightarrow{\mathbf{MN}} \perp \overrightarrow{\mathbf{AB}}$ and $\overrightarrow{\mathbf{MN}} \perp \overrightarrow{\mathbf{CD}}$.

Problem 1.11.2 In a tetrahedron $ABCD$, $\left\|\overrightarrow{\mathbf{AB}}\right\| = \left\|\overrightarrow{\mathbf{BC}}\right\|$, $\left\|\overrightarrow{\mathbf{AD}}\right\| = \left\|\overrightarrow{\mathbf{DC}}\right\|$. Prove that $\overrightarrow{\mathbf{AC}} \perp \overrightarrow{\mathbf{BD}}$.

Problem 1.11.3 In cube $\mathbf{ABCDD'C'B'A'}$ of edge of length a, find the distance between the lines that contain the diagonals $[A'B]$ and $[AC]$.

1.12 Cavalieri, and the Pappus-Guldin Rules

96 Theorem (Cavalieri's Principle) All planar regions with cross sections of proportional length at the same height have area in the same proportion. All solids with cross sections of proportional areas at the same height have their volume in the same proportion.

Proof: *We only provide the proof for the second statement, as the proof for the first is similar. Cut any two such solids by horizontal planes that produce cross sections of area* $A(x)$ *and* $cA(x)$, *where* $c > 0$ *is the constant of proportionality, at an arbitrary height* x *above a fixed base. From elementary calculus, we know that* $\int_{x_1}^{x_2} A(x)\mathrm{d}x$ *and* $\int_{x_1}^{x_2} cA(x)\mathrm{d}x$ *give the volume of the portion of each solid cut by all horizontal planes as* x *runs over some interval* $[x_1; x_2]$. *As* $\int_{x_1}^{x_2} A(x)\mathrm{d}x = c\int_{x_1}^{x_2} A(x)\mathrm{d}x$ *the corresponding volumes must also be proportional.* ❑

97 Example Use Cavalieri's Principle in order to deduce that the area enclosed by the ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b} = 1$, $a > 0, b > 0$, is πab.

Solution: ▶ *Consider the circle with equation $x^2 + y^2 = a^2$, as in figure 1.79. Then, for $y > 0$,*

$$y = \sqrt{a^2 - x^2}, \qquad y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

The corresponding ordinate for the ellipse and the circle are proportional, and hence, the corresponding chords for the ellipse and the circle will be proportional. By Cavalieri's first principle,

$$\begin{aligned}\text{Area of the ellipse} &= \frac{b}{a}\text{Area of the circle}\\ &= \frac{b}{a}\pi a^2\\ &= \pi ab.\end{aligned}$$

◀

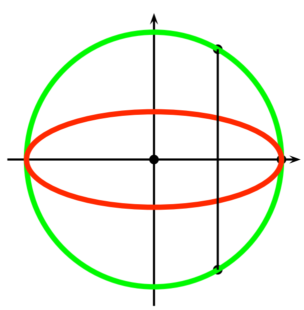

Figure 1.79: Ellipse and circle.

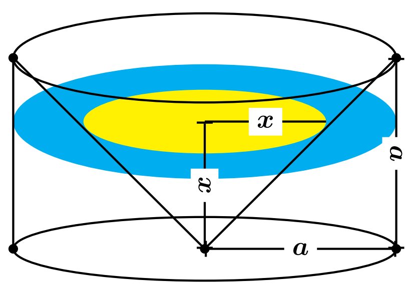

Figure 1.80: Punctured cylinder.

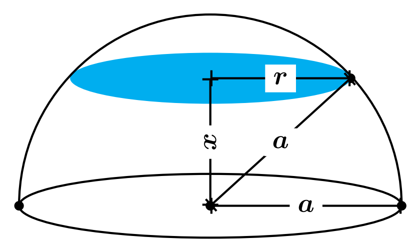

Figure 1.81: Hemisphere.

98 Example Use Cavalieri's Principle in order to deduce that the volume of a sphere with radius a is $\dfrac{4}{3}\pi a^3$.

Solution: ▶ *The following method is due to Archimedes, who was so proud of it that he wanted a sphere inscribed in a cylinder on his tombstone. We need to recall that the volume of a right circular cone with base radius a and height h is $\dfrac{\pi a^2 h}{3}$.*

Consider a hemisphere of radius a, as in figure 1.81. Cut a horizontal slice at height x, producing a circle of radius r. By the Pythagorean Theorem, $x^2 + r^2 = a^2$, and so this circular slab has area $\pi r^2 = \pi(a^2 - x^2)$. Now, consider a punctured cylinder of base radius a and height a, as in figure 1.80, with a cone of height a and base radius a cut from it. A horizontal slab at height x is an annular region of area $\pi a^2 - \pi x^2$, which agrees with a horizontal slab for the sphere at the same height. By Cavalieri's Principle,

$$\begin{aligned}\text{Volume of the hemisphere} &= \text{Volume of the punctured cylinder}\\ &= \pi a^3 - \frac{\pi a^3}{3}\\ &= \frac{2\pi a^3}{3}.\end{aligned}$$

It follows that the volume of the sphere is $2\left(\frac{2\pi a^3}{3}\right) = \frac{4\pi a^3}{3}$. ◄

Essentially the same method of proof as Cavalieri's Principle gives the next result.

99 Theorem (Pappus-Guldin Rule) The area of the lateral surface of a solid of revolution is equal to the product of the length of the generating curve on the side of the axis of revolution and the length of the path described by the centre of gravity of the generating curve under a full revolution. The volume of a solid of revolution is equal to the product of the area of the generating plane on one side of the revolution axis and the length of the path described by the centre of gravity of the area under a full revolution about the axis.

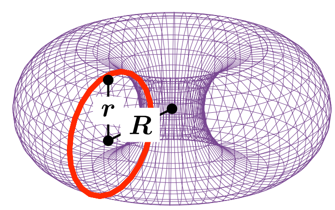

Figure 1.82: A torus.

100 Example Since the centre of gravity of a circle is at its centre, by the Pappus-Guldin Rule, the surface area of the torus with the generating circle having radius r, and radius of gyration R (as in figure 1.82) is $(2\pi r)(2\pi R) = 4\pi^2 rR$. Also, the volume of the solid torus is $(\pi r^2)(2\pi R) = 2\pi^2 r^2 R$.

Homework

Problem 1.12.1 Use the Pappus-Guldin Rule to find the lateral area and the volume of a right circular cone with base radius r and height h.

1.13 Dihedral Angles and Platonic Solids

101 Definition When two half planes intersect in space they intersect on a line. The portion of space bounded by the half planes and the line is called the *dihedral angle*. The intersecting line is called the *edge* of the dihedral angle and each of the two half planes of the dihedral angle is called a *face*. See figure 1.83.

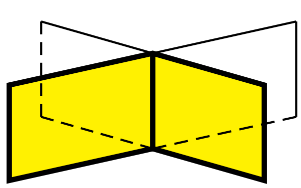

Figure 1.83: Dihedral Angles.

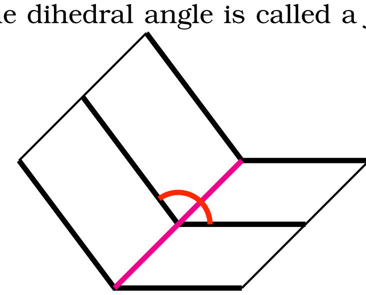

Figure 1.84: Rectilinear of a Dihedral Angle.

102 Definition The *rectilinear angle* of a dihedral angle is the angle whose sides are perpendicular to the edge of the dihedral angle at the same point, each on each of the faces. See figure 1.84.

All the rectilinear angles of a dihedral angle measure the same. Hence the *measure* of a dihedral angle is the measure of any one of its rectilinear angles.

In analogy to dihedral angles we now define polyhedral angles.

103 Definition The opening of three or more planes that meet at a common point is called a *polyhedral angle* or *solid angle*. In the particular case of three planes, we use the term *trihedral angle*. The common point is called the *vertex* of the polyhedral angle. Each of the intersecting lines of two consecutive planes is called an *edge* of the polyhedral angle. The portion of the planes lying between consecutive edges are called the *faces* of the polyhedral angle. The angles formed by adjacent edges are called *face angles*. A polyhedral angle is said to be *convex* if the section made by a plane cutting all its edges forms a convex polygon.

In the trihedral angle of figure 1.85, V is the vertex, $\triangle VAB$, $\triangle VBC$, $\triangle VCA$ are faces. Also, notice that in any polyhedral angle, any two adjacent faces form a dihedral angle.

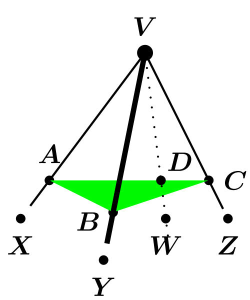

Figure 1.85: Trihedral Angle.

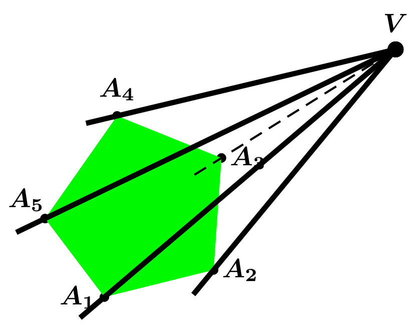

Figure 1.86: Polyhedral Angle.

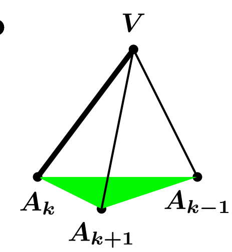

Figure 1.87: A, A_k, A_{k+1} are three consecutive vertices.

104 Theorem The sum of any two face angles of a trihedral angle is greater than the third face angle.

Proof: *Consider figure 1.85. If $\angle ZVX$ is smaller or equal to in size than either $\angle XVY$ or YVZ, then we are done, so assume that, say, $\angle ZVX > XVY$. We must demonstrate that*

$$\angle XVY + \angle YVZ > \angle ZVX.$$

Since we are assuming that $\angle ZVX > XVY$, we may draw, in $\angle XVY$ the line segment $[VW]$ such that $\angle XVW = \angle XVY$.

Through any point D of the segment $[VW]$, draw $\triangle ADC$ on the plane P containing the points V, X, Z. Take the point $B \in [VY]$ so that $VD = VB$. Consider now the plane containing the line segment $[AC]$ and the point B. Observe that $\triangle AVD \cong AVB$. Hence $AD = AB$. Now, by the triangle inequality in $\triangle ABC$, $AB + BC > CA$. This implies that $\angle BVC > \angle DVC$. Hence

$$\begin{aligned} \angle AVB + \angle BVC &= \angle AVD + \angle BVC \\ &> \angle AVD + \angle DVC \\ &= \angle AVC, \end{aligned}$$

which proves that $\angle XVY + \angle YVZ > \angle ZVX$, as wanted. ❑

105 Theorem The sum of the face angles of any convex polyhedral angle is less than 2π radians.

Proof: *Let the polyhedral angle have n faces and vertex V. Let the faces be cut by a plane, intersecting the edges at the points $A_1, A_2, \ldots A_n$, say. An illustration can be seen in figure 1.86, where for convenience, we have depicted only five edges. Observe that the polygon $A_1A_2\cdots A_n$ is convex and that the sum of its interior angles is $\pi(n-2)$. We would like to prove that*

$$\angle A_1VA_2 + \angle A_2VA_3 + \angle A_3VA_4 + \cdots + \angle A_{n-1}VA_n + \angle A_nVA_1 < 2\pi.$$

Now, let A_{k-1}, A_k, A_{k+1} be three consecutive vertices of of the polygon $A_1A_2\cdots A_n$. This notation means that $A_{k-1}A_kA_{k+1}$ represents any of the n triplets $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_5, \ldots,$ $A_{n-2}A_{n-1}A_n$, $A_{n-1}A_nA_1$, $A_nA_1A_2$, that is, we let $A_0 = A_n$, $A_{n+1} = A_1$, $A_{n+2} = A_2$, etc. Consider the trihedral angle with vertex A_k and whose face angles at A_k are $\angle A_{k-1}A_kA_{k+1}$, $\angle VA_kA_{k-1}$, and $\angle VA_kA_{k+1}$, as in figure 1.87. Observe that as k ranges from 1 through n, the sum

$$\sum_{1\leq k\leq n} \angle A_{k-1}A_kA_{k+1} = \pi(n-2),$$

being the sum of the interior angles of the polygon $A_1A_2\cdots A_n$. By Theorem 104,

$$\angle VA_kA_{k-1} + \angle VA_kA_{k+1} > \angle A_{k-1}A_kA_{k+1}.$$

Thus

$$\sum_{1\leq k\leq n} VA_kA_{k-1} + \angle VA_kA_{k+1} > \sum_{1\leq k\leq n} \angle A_{k-1}A_kA_{k+1} = \pi(n-2).$$

Also,

$$\sum_{1\leq k\leq n} VA_kA_{k+1} + \angle VA_{k+1}A_k + \angle A_kVA_{k+1} = \pi n,$$

since this is summing the sum of the angles of the n triangles of the faces. But clearly

$$\sum_{1\leq k\leq n} VA_kA_{k+1} = \sum_{1\leq k\leq n} \angle VA_{k+1}A_k,$$

since one sum adds the angles in one direction and the other in the opposite direction. For the same reason,

$$\sum_{1\leq k\leq n} VA_kA_{k-1} = \sum_{1\leq k\leq n} \angle VA_kA_{k+1}.$$

Hence

$$\begin{aligned}\sum_{1\leq k\leq n} \angle A_kVA_{k+1} &= \pi n - \sum_{1\leq k\leq n} (VA_kA_{k+1} + \angle VA_{k+1}A_k) \\ &= \pi n - \sum_{1\leq k\leq n} (VA_kA_{k+1} + \angle VA_kA_{k-1}) \\ &< \pi n - \pi(n-2) \\ &= 2\pi,\end{aligned}$$

as we needed to shew. ❑

106 Definition A *Platonic solid* is a polyhedron having congruent regular polygon as faces and having the same number of edges meeting at each corner.

Suppose a regular polygon with $n \geq 3$ sides is a face of a platonic solid with $m \geq 3$ faces meeting at a corner. Since each interior angle of this polygon measures $\dfrac{\pi(n-2)}{n}$, we must have in view of Theorem 105,

$$m\left(\frac{\pi(n-2)}{n}\right) < 2\pi \implies m(n-2) < 2n \implies (m-2)(n-2) < 4.$$

Since $n \geq 3$ and $m \geq 3$, the above inequality only holds for five pairs (n, m). Appealing to Euler's Formula for polyhedrons, which states that $V + F = E + 2$, where V is the number of vertices, F is the number of faces, and E is the number of edges of a polyhedron, we obtain the values in the following table.

m	n	S	E	F	Name of regular Polyhedron.
3	3	4	6	4	Tetrahedron or regular Pyramid.
4	3	8	12	6	Hexahedron or Cube.
3	4	6	12	8	Octahedron.
5	3	20	30	12	Dodecahedron.
3	5	12	30	20	Icosahedron.

Thus there are at most five Platonic solids. That there are exactly five can be seen by explicit construction. Figures 1.88 through 1.92 depict the Platonic solids.

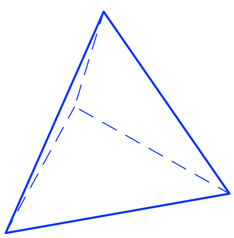

Figure 1.88: Tetrahedron.

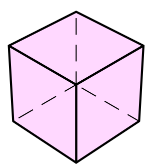

Figure 1.89: Cube or hexahedron.

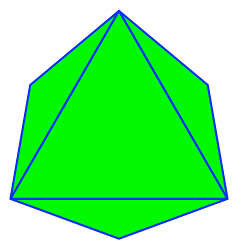

Figure 1.90: Octahedron.

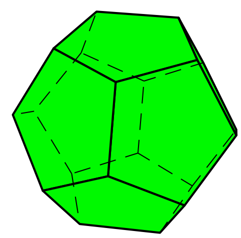

Figure 1.91: Dodecahedron

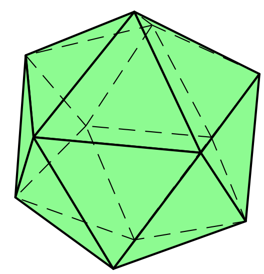

Figure 1.92: Icosahedron.

1.14 Spherical Trigonometry

Consider a point $B(x, y, z)$ in Cartesian coordinates. From $O(0, 0, 0)$ we draw a straight line to $B(x, y, z)$, and let its distance be ρ. We measure its inclination from the positive z-axis, let us say it is an angle of ϕ, $\phi \in [0; \pi]$ radians, as in figure 1.93. Observe that $z = \rho \cos \phi$. We now project the line segment $[OB]$ onto the xy-plane in order to find the polar coordinates of x and y. Let θ be angle that this projection makes with the positive x-axis. Since $OP = \rho \sin \phi$ we find $x = \rho \cos \theta \sin \phi$, $y = \rho \sin \theta \sin \phi$.

107 Definition Given a point (x, y, z) in Cartesian coordinates, its *spherical coordinates* are given by

$$x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, z = \rho \cos \phi.$$

Here ϕ is the *polar angle*, measured from the positive z-axis, and θ is the *azimuthal angle*, measured from the positive x-axis. By convention, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

Spherical coordinates are extremely useful when considering regions which are symmetric about a point.

108 Definition If a plane intersects with a sphere, the intersection will be a circle. If this circle contains the centre of the sphere, we call it a *great circle*. Otherwise we talk of a *small circle*. The *axis* of any circle on a sphere is the diameter of the sphere which is normal to the plane containing the circle. The endpoints of such a diameter are called the *poles* of the circle.

☞ *The radius of a great circle is the radius of the sphere. The poles of a great circle are equally distant from the plane of the circle, but this is not the case in a small circle. By* the pole *of a small circle, we mean the closest pole to the plane containing the circle. A pole of a circle is equidistant from every point of the circumference of the circle.*

109 Definition Given the centre of the sphere, and any two points of the surface of the sphere, a plane can be drawn. This plane will be unique if and only if the points are not diametrically opposite. In the case where the two points are not diametrically opposite, the great circle formed is split into a larger and a smaller arc by the two points. We call the smaller arc the *geodesic* joining the two points. If the two points are diametrically opposite then every plane containing the line forms with the sphere a great circle, and the arcs formed are then of equal length. In this case we take any such arc as a geodesic.

110 Definition A *spherical triangle* is a triangle on the surface of a sphere all whose vertices are connected by geodesics. The three arcs of great circles which form a spherical triangle are called the *sides* of the spherical triangle; the angles formed by the arcs at the points where they meet are called the *angles* of the spherical triangle.

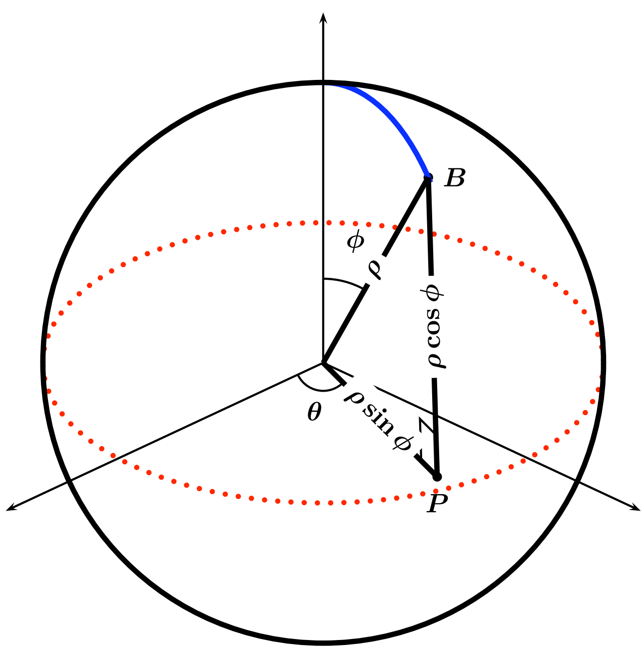

Figure 1.93: Spherical Coordinates.

If A, B, C are the vertices of a spherical triangle, it is customary to label the opposite arcs with the same letter name, but in lowercase.

☞ *A spherical triangle has then six angles: three vertex angles $\angle A$, $\angle B$, $\angle C$, and three arc angles, $\angle a$, $\angle b$, $\angle c$. Observe that if O is the centre of the sphere then*

$$\angle a = \angle\left(\overrightarrow{\mathbf{OB}}, \overrightarrow{\mathbf{OC}}\right), \quad \angle b = \angle\left(\overrightarrow{\mathbf{OC}}, \overrightarrow{\mathbf{OA}}\right), \quad \angle c = \angle\left(\overrightarrow{\mathbf{OA}}, \overrightarrow{\mathbf{OB}}\right),$$

and

$$\angle A = \angle\left(\overrightarrow{\mathbf{OA}} \times \overrightarrow{\mathbf{OB}}, \overrightarrow{\mathbf{OA}} \times \overrightarrow{\mathbf{OC}}\right), \quad \angle B = \angle\left(\overrightarrow{\mathbf{OB}} \times \overrightarrow{\mathbf{OC}}, \overrightarrow{\mathbf{OB}} \times \overrightarrow{\mathbf{OA}}\right), \quad \angle C = \angle\left(\overrightarrow{\mathbf{OC}} \times \overrightarrow{\mathbf{OA}}, \overrightarrow{\mathbf{OC}} \times \overrightarrow{\mathbf{OB}}\right).$$

111 Theorem Let $\triangle ABC$ be a spherical triangle. Then

$$\cos a \cos b + \sin a \sin b \cos C = \cos c.$$

Proof: *Consider a spherical triangle ABC with $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and let O be the centre and ρ be the radius of the sphere. In spherical coordinates this is, say,*

$$\begin{array}{lll} z_1 = \rho\cos\theta_1, & x_1 = \rho\sin\theta_1\cos\phi_1, & y_1 = \rho\sin\theta_1\sin\phi_1, \\ z_2 = \rho\cos\theta_2, & x_2 = \rho\sin\theta_2\cos\phi_2, & y_2 = \rho\sin\theta_2\sin\phi_2; \end{array}$$

By a rotation we may assume that the z-axis passes through C. Then the following quantities give the square of the distance of the line segment $[AB]$:

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2, \quad \rho^2 + \rho^2 - 2\rho^2\cos\angle(AOB).$$

Since $x_1^2 + y_1^2 + z_1^2 = \rho^2$, $x_2^2 + y_2^2 + z_2^2 = \rho^2$, we gather that

$$x_1x_2 + y_1y_2 + z_1z_2 = \rho^2\cos\angle(AOB).$$

Therefore we obtain

$$\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1\cos(\phi_1 - \phi_2) = \cos\angle(AOB),$$

that is,

$$\cos a\cos b + \sin a\sin b\cos C = \cos c.$$

❑

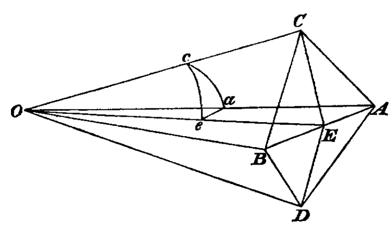

Figure 1.94: Theorem 112.

112 Theorem Let I be the dihedral angle of two adjacent faces of a regular polyhedron. Then

$$\sin\frac{I}{2} = \frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{m}}.$$

Proof: *See figure 1.94. Let AB be the edge common to the two adjacent faces, C and D the centres of the faces; bisect AB at E, and join CE and DE; CE and DE will be perpendicular to AB, and the angle CED is the angle of inclination of the two adjacent faces; we shall denote it by I. In the plane containing CE and DE draw CO and DO at right angles to CE and DE respectively, and meeting at O; about O as centre describe a sphere meeting OA, OC, OE at a, c, e respectively, so that cae forms a spherical triangle. Since AB is perpendicular to CE and DE, it is perpendicular to the plane CED, therefore the plane AOB which contains AB is perpendicular to the plane CED; hence the angle cea of the spherical triangle is a right angle. Let m be the number of sides in each face of the polyhedron, n the number of the plane angles which form each solid angle. Then the angle $ace = ACE = \dfrac{2\pi}{2m} = \dfrac{\pi}{m}$; and the angle cae is half one of the n equal angles formed on the sphere round a, that is, $cae = \dfrac{2\pi}{2n} = \dfrac{\pi}{n}$. From the right-angled triangle cae*

$$\cos cae = \cos cOe\sin ace,$$

that is $$\cos\frac{\pi}{n} = \cos\left(\frac{\pi}{2} - \frac{I}{2}\right)\sin\frac{\pi}{m};$$

therefore $$\sin\frac{I}{2} = \frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{m}}.$$

❑

113 Theorem Let r and R be, respectively, the radii of the inscribed and circumscribed spheres of a regular polyhedron. Then
$$r = \frac{a}{2}\cot\frac{\pi}{m}\tan\frac{I}{2}, \qquad R = \frac{a}{2}\tan\frac{I}{2}\tan\frac{\pi}{n}.$$
Here a is the length of any edge of the polyhedron, and I is the dihedral angle of any two faces.

Proof: *Let the edge $AB = a$, let $OC = r$ and $OA = R$, so that r is the radius of the inscribed sphere, and R is the radius of the circumscribed sphere. Then*

$$CE = AE\cot ACE = \frac{a}{2}\cot\frac{\pi}{m},$$

$$r = CE\tan CEO = CE\tan\frac{I}{2} = \frac{a}{2}\cot\frac{\pi}{m}\tan\frac{I}{2};$$

also $$r = R\cos aOc = R\cot eca\cot eac = R\cot\frac{\pi}{m}\cot\frac{\pi}{n};$$

therefore $$R = r\tan\frac{\pi}{m}\tan\frac{\pi}{n} = \frac{a}{2}\tan\frac{I}{2}\tan\frac{\pi}{n}.$$

❑

From the above formulæ we now easily find that the volume of the pyramid which has one face of the polyhedron for base and O for vertex is $\frac{r}{3}\cdot\frac{ma^2}{4}\cot\frac{\pi}{m}$, and therefore the volume of the polyhedron is $\frac{mFra^2}{12}\cot\frac{\pi}{m}$.

Furthermore, the area of one face of the polyhedron is $\frac{ma^2}{4}\cot\frac{\pi}{m}$, and therefore the surface area of the polyhedron is $\frac{mFa^2}{4}\cot\frac{\pi}{m}$.

Homework

Problem 1.14.1 The four vertices of a regular tetrahedron are

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix}.$$

What is the cosine of the dihedral angle between any pair of faces of the tetrahedron?

Problem 1.14.2 Consider a tetrahedron whose edge measures a. Shew that its volume is $\frac{a^3\sqrt{2}}{12}$, its surface area is $a^2\sqrt{3}$, and that the radius of the inscribed sphere is $\frac{a\sqrt{6}}{12}$.

Problem 1.14.3 Consider a cube whose edge measures a. Shew that its volume is a^3, its surface area is $6a^2$, and that the radius of the inscribed sphere is $\frac{a}{2}$.

Problem 1.14.4 Consider an octahedron whose edge measures a. Shew that its volume is $\frac{a^3\sqrt{2}}{3}$, its surface area is $2a^2\sqrt{3}$, and that the radius of the inscribed sphere is $\frac{a\sqrt{6}}{6}$.

Problem 1.14.5 Consider a dodecahedron whose edge measures a. Shew that its volume is $\frac{a^3}{4}\left(15+7\sqrt{5}\right)$, its surface area is $3a^2\sqrt{25+10\sqrt{5}}$, and that the radius of the inscribed sphere is $\frac{a}{4}\sqrt{10+22\sqrt{\frac{1}{5}}}$.

Problem 1.14.6 Consider an icosahedron whose edge measures a. Shew that its volume is $\frac{5a^3}{12}\left(3+\sqrt{5}\right)$, its surface area is $5a^2\sqrt{3}$, and that the radius of the inscribed sphere is $\frac{a}{12}\left(5\sqrt{3}+\sqrt{15}\right)$.

1.15 Canonical Surfaces

In this section we consider various surfaces that we shall periodically encounter in subsequent sections. Just like in one-variable Calculus it is important to identify the equation and the shape of a line, a parabola, a circle, etc., it will become important for us to be able to identify certain families of often-occurring surfaces. We shall explore both their Cartesian and their parametric form. We remark that in order to parametrise curves ("one-dimensional entities") we needed one parameter, and that in order to parametrise surfaces we shall need to parameters.

Let us start with the plane. Recall that if a, b, c are real numbers, not all zero, then the Cartesian equation of a plane with normal vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and passing through the point (x_0, y_0, z_0) is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

If we know that the vectors $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ are on the plane (parallel to the plane) then with the parameters $p, q$the equation of the plane is

$$\begin{aligned} x - x_0 &= pu_1 + qv_1, \\ y - y_0 &= pu_2 + qv_2, \\ z - z_0 &= pu_3 + qv_3. \end{aligned}$$

114 Definition A surface S consisting of all lines parallel to a given line Δ and passing through a given curve Γ is called a *cylinder*. The line Δ is called the *directrix* of the cylinder.

☞ *To recognise whether a given surface is a cylinder we look at its Cartesian equation. If it is of the form $f(A, B) = 0$, where A, B are secant planes, then the curve is a cylinder. Under these conditions, the lines generating S will be parallel to the line of equation $A = 0, B = 0$. In practice, if one of the variables x, y, or z is missing, then the surface is a cylinder, whose directrix will be the axis of the missing coordinate.*

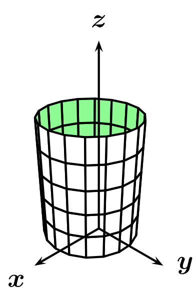

Figure 1.95: Circular cylinder $x^2 + y^2 = 1$.

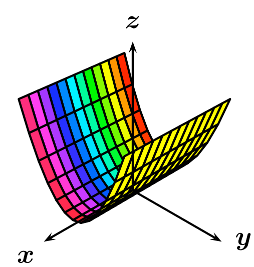

Figure 1.96: The parabolic cylinder $z = y^2$.

115 Example Figure 1.95 shews the cylinder with Cartesian equation $x^2 + y^2 = 1$. One starts with the circle $x^2 + y^2 = 1$ on the xy-plane and moves it up and down the z-axis. A parametrisation for this cylinder is the following:

$$x = \cos v, \qquad y = \sin v, \qquad z = u, \qquad u \in \mathbb{R}, v \in [0; 2\pi].$$

The Maple™ commands to graph this surface are:

```
> with(plots):
> implicitplot3d(x^2+y^2=1,x=-1..1,y=-1..1,z=-10..10);
> plot3d([cos(s),sin(s),t],s=-10..10,t=-10..10,numpoints=5001);
```

The method of parametrisation utilised above for the cylinder is quite useful when doing parametrisations in space. We refer to it as the method of *cylindrical coordinates*. In general, we first find the polar coordinates of x, y in the xy-plane, and then lift $(x, y, 0)$ parallel to the z-axis to (x, y, z):

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

See figure 1.97.

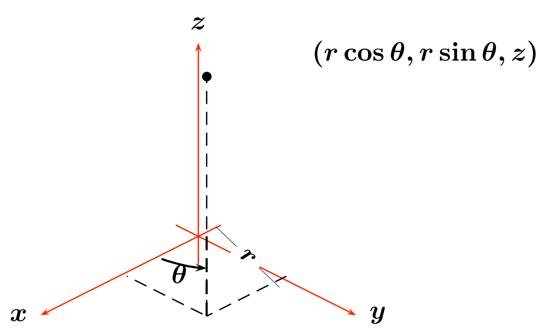

Figure 1.97: Cylindrical Coordinates.

116 Example Figure 1.96 shews the parabolic cylinder with Cartesian equation $z = y^2$. One starts with the parabola $z = y^2$ on the yz-plane and moves it up and down the x-axis. A parametrisation for this parabolic cylinder is the following:

$$x = u, \qquad y = v, \qquad z = v^2, \qquad u \in \mathbb{R}, v \in \mathbb{R}.$$

The Maple™ commands to graph this surface are:

```
> with(plots):
> implicitplot3d(z=y^2,x=-10..10,y=-10..10,z=-10..10,numpoints=5001);
> plot3d([t,s,s^2],s=-10..10,t=-10..10,numpoints=5001,axes=boxed);
```

117 Example Figure 1.98 shews the hyperbolic cylinder with Cartesian equation $x^2 - y^2 = 1$. One starts with the hyperbola $x^2 - y^2$ on the xy-plane and moves it up and down the z-axis. A parametrisation for this parabolic cylinder is the following:

$$x = \pm\cosh v, \qquad y = \sinh v, \qquad z = u, \qquad u \in \mathbb{R}, v \in \mathbb{R}.$$

We need a choice of sign for each of the portions. We have used the fact that $\cosh^2 v - \sinh^2 v = 1$. The Maple™ commands to graph this surface are:

```
> with(plots):
> implicitplot3d(x^2-y^2=1,x=-10..10,y=-10..10,z=-10..10,numpoints=5001);
> plot3d({[-cosh(s),sinh(s),t],[cosh(s),sinh(s),t]},
> s=-2..2,t=-10..10,numpoints=5001,axes=boxed);
```

118 Definition Given a point $\Omega \in \mathbb{R}^3$ (called the *apex*) and a curve Γ (called the generating curve), the surface S obtained by drawing rays from Ω and passing through Γ is called a *cone*.

☞ *In practice, if the Cartesian equation of a surface can be put into the form* $f(\frac{A}{C}, \frac{B}{C}) = 0$, *where* A, B, C, *are planes secant at exactly one point, then the surface is a cone, and its apex is given by* $A = 0, B = 0, C = 0$.

119 Example The surface in $\mathbb{R}^3$ implicitly given by

$$z^2 = x^2 + y^2$$

is a cone, as its equation can be put in the form $\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1 = 0$. Considering the planes $x = 0, y = 0, z = 0$, the apex is located at $(0, 0, 0)$. The graph is shewn in figure 1.100.

120 Definition A surface S obtained by making a curve Γ turn around a line Δ is called a *surface of revolution*. We then say that Δ is the axis of revolution. The intersection of S with a half-plane bounded by Δ is called a *meridian*.

☞ *If the Cartesian equation of* S *can be put in the form* $f(A, \Sigma) = 0$, *where* A *is a plane and* Σ *is a sphere, then the surface is of revolution. The axis of* S *is the line passing through the centre of* Σ *and perpendicular to the plane* A.

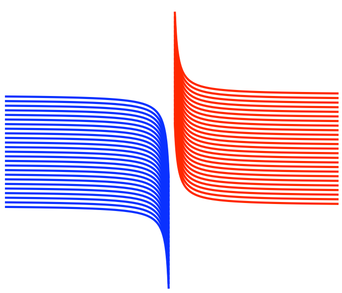

Figure 1.98: The hyperbolic cylinder $x^2 - y^2 = 1$.

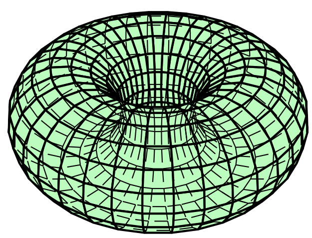

Figure 1.99: The torus.

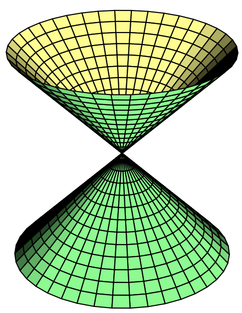

Figure 1.100: Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$.

121 Example Find the equation of the surface of revolution generated by revolving the hyperbola

$$x^2 - 4z^2 = 1$$

about the z-axis.

Solution: ▶ *Let* (x, y, z) *be a point on* S. *If this point were on the* xz *plane, it would be on the hyperbola, and its distance to the axis of rotation would be* $|x| = \sqrt{1 + 4z^2}$. *Anywhere else, the distance of* (x, y, z) *to the axis of rotation is the same as the distance of* (x, y, z) *to* $(0, 0, z)$, *that is* $\sqrt{x^2 + y^2}$. *We must have*

$$\sqrt{x^2 + y^2} = \sqrt{1 + 4z^2},$$

which is to say

$$x^2 + y^2 - 4z^2 = 1.$$

This surface is called a hyperboloid of one sheet. *See figure 1.104. Observe that when* $z = 0$, $x^2 + y^2 = 1$ *is a circle on the* xy *plane. When* $x = 0$, $y^2 - 4z^2 = 1$ *is a hyperbola on the* yz *plane. When* $y = 0$, $x^2 - 4z^2 = 1$ *is a hyperbola on the* xz *plane.*

A parametrisation for this hyperboloid is

$$x = \sqrt{1+4u^2}\cos v, \qquad y = \sqrt{1+4u^2}\sin v, \qquad z = u, \qquad u \in \mathbb{R}, v \in [0; 2\pi].$$

◀

122 Example The circle $(y-a)^2 + z^2 = r^2$, on the yz plane (a, r are positive real numbers) is revolved around the z-axis, forming a torus T. Find the equation of this torus.

Solution: ▶ *Let (x, y, z) be a point on T. If this point were on the yz plane, it would be on the circle, and the of the distance to the axis of rotation would be $y = a + \operatorname{sgn}(y-a)\sqrt{r^2 - z^2}$, where $\operatorname{sgn}(t)$ (with $\operatorname{sgn}(t) = -1$ if $t < 0$, $\operatorname{sgn}(t) = 1$ if $t > 0$, and $\operatorname{sgn}(0) = 0$) is the sign of t. Anywhere else, the distance from (x, y, z) to the z-axis is the distance of this point to the point (x, y, z) : $\sqrt{x^2 + y^2}$. We must have*

$$x^2 + y^2 = (a + \operatorname{sgn}(y-a)\sqrt{r^2 - z^2})^2 = a^2 + 2a\operatorname{sgn}(y-a)\sqrt{r^2 - z^2} + r^2 - z^2.$$

Rearranging

$$x^2 + y^2 + z^2 - a^2 - r^2 = 2a\operatorname{sgn}(y-a)\sqrt{r^2 - z^2},$$

or

$$(x^2 + y^2 + z^2 - (a^2 + r^2))^2 = 4a^2r^2 - 4a^2z^2$$

since $(\operatorname{sgn}(y-a))^2 = 1$, (it could not be 0, why?). Rearranging again,

$$(x^2 + y^2 + z^2)^2 - 2(a^2 + r^2)(x^2 + y^2) + 2(a^2 - r^2)z^2 + (a^2 - r^2)^2 = 0.$$

The equation of the torus thus, is of fourth degree, and its graph appears in figure 1.99.

A parametrisation for the torus generated by revolving the circle $(y-a)^2 + z^2 = r^2$ around the z-axis is

$$x = a\cos\theta + r\cos\theta\cos\alpha, \qquad y = a\sin\theta + r\sin\theta\cos\alpha, \qquad z = r\sin\alpha,$$

with $(\theta, \alpha) \in [-\pi; \pi]^2$.

◀

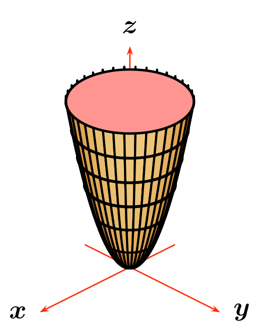

Figure 1.101: Paraboloid $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$.

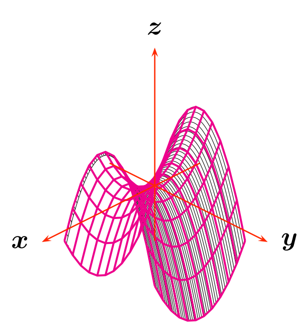

Figure 1.102: Hyperbolic paraboloid $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

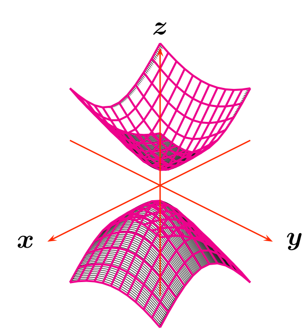

Figure 1.103: Two-sheet hyperboloid $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + 1$.

123 Example The surface $z = x^2 + y^2$ is called an *elliptic paraboloid*. The equation clearly requires that $z \geq 0$. For fixed $z = c$, $c > 0$, $x^2 + y^2 = c$ is a circle. When $y = 0$, $z = x^2$ is a parabola on the xz plane. When $x = 0$, $z = y^2$ is a parabola on the yz plane. See figure 1.101. The following is a parametrisation of this paraboloid:

$$x = \sqrt{u}\cos v, \qquad y = \sqrt{u}\sin v, \qquad z = u, \qquad u \in [0; +\infty[, v \in [0; 2\pi].$$

124 Example The surface $z = x^2 - y^2$ is called a *hyperbolic paraboloid* or *saddle*. If $z = 0$, $x^2 - y^2 = 0$ is a pair of lines in the xy plane. When $y = 0$, $z = x^2$ is a parabola on the xz plane. When $x = 0$, $z = -y^2$ is a parabola on the yz plane. See figure 1.102. The following is a parametrisation of this hyperbolic paraboloid:

$$x = u, \qquad y = v, \qquad z = u^2 - v^2, \qquad u \in \mathbb{R}, v \in \mathbb{R}.$$

125 Example The surface $z^2 = x^2 + y^2 + 1$ is called an *hyperboloid of two sheets*. For $z^2 - 1 < 0$, $x^2 + y^2 < 0$ is impossible, and hence there is no graph when $-1 < z < 1$. When $y = 0$, $z^2 - x^2 = 1$ is a hyperbola on the xz plane. When $x = 0$, $z^2 - y^2 = 1$ is a hyperbola on the yz plane. When $z = c$ is a constant $c > 1$, then the $x^2 + y^2 = c^2 - 1$ are circles. See figure 1.103. The following is a parametrisation for the top sheet of this hyperboloid of two sheets

$$x = u\cos v, \qquad y = u\sin v, \qquad z = u^2 + 1, \qquad u \in \mathbb{R}, v \in [0; 2\pi]$$

and the following parametrises the bottom sheet,

$$x = u\cos v, \qquad y = u\sin v, \qquad z = -u^2 - 1, \qquad u \in \mathbb{R}, v \in [0; 2\pi],$$

126 Example The surface $z^2 = x^2 + y^2 - 1$ is called an *hyperboloid of one sheet*. For $x^2 + y^2 < 1$, $z^2 < 0$ is impossible, and hence there is no graph when $x^2 + y^2 < 1$. When $y = 0$, $z^2 - x^2 = -1$ is a hyperbola on the xz plane. When $x = 0$, $z^2 - y^2 = -1$ is a hyperbola on the yz plane. When $z = c$ is a constant, then the $x^2 + y^2 = c^2 + 1$ are circles See figure 1.104. The following is a parametrisation for this hyperboloid of one sheet

$$x = \sqrt{u^2 + 1}\cos v, \qquad y = \sqrt{u^2 + 1}\sin v, \qquad z = u, \qquad u \in \mathbb{R}, v \in [0; 2\pi],$$

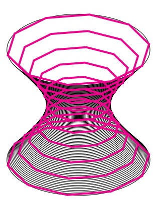

Figure 1.104: One-sheet hyperboloid $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$.

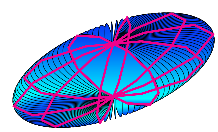

Figure 1.105: Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

127 Example Let a, b, c be strictly positive real numbers. The surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is called an *ellipsoid*. For $z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2}1$ is an ellipse on the xy plane.When $y = 0$, $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ is an ellipse on the xz plane. When $x = 0$, $\frac{z^2}{c^2} + \frac{y^2}{b^2} = 1$ is an ellipse on the yz plane. See figure 1.105. We may parametrise the ellipsoid using spherical coordinates:

$$x = a\cos\theta\sin\phi, \qquad y = b\sin\theta\sin\phi, \qquad z = c\cos\phi, \qquad \theta \in [0; 2\pi], \phi \in [0; \pi].$$

Homework

Problem 1.15.1 Find the equation of the surface of revolution S generated by revolving the ellipse $4x^2 + z^2 = 1$ about the z-axis.

Problem 1.15.2 Find the equation of the surface of revolution generated by revolving the line $3x + 4y = 1$ about the y-axis .

Problem 1.15.3 Describe the surface parametrised by $\varphi(u, v) \mapsto (v \cos u, v \sin u, au)$, $(u, v) \in (0, 2\pi) \times (0, 1)$, $a > 0$.

Problem 1.15.4 Describe the surface parametrised by $\varphi(u, v) = (au \cos v, bu \sin v, u^2)$, $(u, v) \in (1, +\infty) \times (0, 2\pi)$, $a, b > 0$.

Problem 1.15.5 Consider the spherical cap defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 1/\sqrt{2}\}.$$

Parametrise S using Cartesian, Spherical, and Cylindrical coordinates.

Problem 1.15.6 Demonstrate that the surface in $\mathbb{R}^3$

$$S : e^{x^2+y^2+z^2} - (x + z)e^{-2xz} = 0,$$

implicitly defined, is a cylinder.

Problem 1.15.7 Shew that the surface in $\mathbb{R}^3$ implicitly defined by

$$x^4 + y^4 + z^4 - 4xyz(x + y + z) = 1$$

is a surface of revolution, and find its axis of revolution.

Problem 1.15.8 Shew that the surface S in $\mathbb{R}^3$ given implicitly by the equation

$$\frac{1}{x-y} + \frac{1}{y-z} + \frac{1}{z-x} = 1$$

is a cylinder and find the direction of its directrix.

Problem 1.15.9 Shew that the surface S in $\mathbb{R}^3$ implicitly defined as

$$xy + yz + zx + x + y + z + 1 = 0$$

is of revolution and find its axis.

Problem 1.15.10 Demonstrate that the surface in $\mathbb{R}^3$ given implicitly by

$$z^2 - xy = 2z - 1$$

is a cone

Problem 1.15.11 (Putnam Exam 1970) Determine, with proof, the radius of the largest circle which can lie on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad a > b > c > 0.$$

Problem 1.15.12 The hyperboloid of one sheet in figure 1.106 has the property that if it is cut by planes at $z = \pm 2$, its projection on the xy plane produces the ellipse $x^2 + \frac{y^2}{4} = 1$, and if it is cut by a plane at $z = 0$, its projection on the xy plane produces the ellipse $4x^2 + y^2 = 1$. Find its equation.

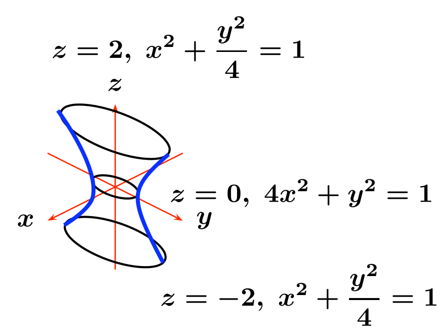

Figure 1.106: Problem 1.15.12.

1.16 Parametric Curves in Space

In analogy to curves on the plane, we now define curves in space.

128 Definition Let $[a; b] \subseteq \mathbb{R}$. A *parametric curve* representation $\mathbf{r}$ of a curve Γ is a function $\mathbf{r} : [a; b] \to \mathbb{R}^3$, with

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},$$

and such that $\mathbf{r}([a;b]) = \Gamma$. $\mathbf{r}(a)$ is the *initial point* of the curve and $\mathbf{r}(b)$ its *terminal point*. A curve is *closed* if its initial point and its final point coincide. The *trace* of the curve $\mathbf{r}$ is the set of all images of $\mathbf{r}$, that is, Γ. The length of the curve is

$$\int_{\Gamma} \left|\left| \mathrm{d}\overrightarrow{\mathbf{r}} \right|\right|.$$

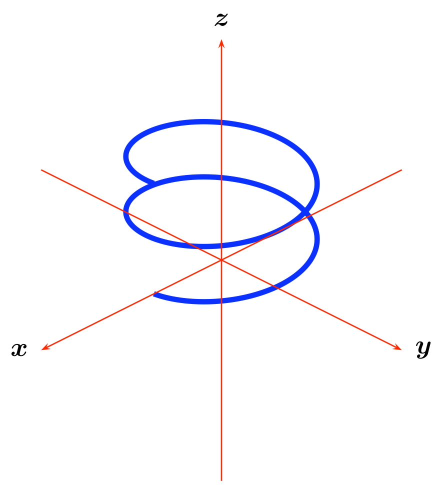

Figure 1.107: Helix.

129 Example The trace of

$$\mathbf{r}(t) = \overrightarrow{\mathbf{i}}\cos t + \overrightarrow{\mathbf{j}}\sin t + \overrightarrow{\mathbf{k}}\,t$$

is known as a *cylindrical helix.* To find the length of the helix as t traverses the interval $[0;2\pi]$, first observe that

$$\left|\left| \mathrm{d}\overrightarrow{\mathbf{x}} \right|\right| = \left|\left| (\sin t)^2 + (-\cos t)^2 + 1 \right|\right| \mathrm{d}t = \sqrt{2}\mathrm{d}t,$$

and thus the length is

$$\int_0^{2\pi} \sqrt{2}\mathrm{d}t = 2\pi\sqrt{2}.$$

The Maple™ commands to graph this curve and to find its length are:

```
>   with(plots):
>   with(Student[VectorCalculus]):
>   spacecurve([cos(t),sin(t),t],t=0..2*Pi,axes=normal);
>   PathInt(1,[x,y,z]=Path(<cos(t),sin(t),t>,0..2*Pi));
```

130 Example Find a parametric representation for the curve resulting by the intersection of the plane $3x + y + z = 1$ and the cylinder $x^2 + 2y^2 = 1$ in $\mathbb{R}^3$.

Solution: ► *The projection of the intersection of the plane $3x + y + z = 1$ and the cylinder is the ellipse $x^2 + 2y^2 = 1$, on the xy-plane. This ellipse can be parametrised as*

$$x = \cos t, \ y = \frac{\sqrt{2}}{2}\sin t, \qquad 0 \le t \le 2\pi.$$

From the equation of the plane,

$$z = 1 - 3x - y = 1 - 3\cos t - \frac{\sqrt{2}}{2}\sin t.$$

Thus we may take the parametrisation

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \frac{\sqrt{2}}{2}\sin t \\ 1 - 3\cos t - \frac{\sqrt{2}}{2}\sin t \end{bmatrix}.$$

◀

131 Example Let a, b, c be strictly positive real numbers. Consider the the region

$$\mathscr{R} = \left\{(x, y, z) \in \mathbb{R}^3 : |x| \le a, |y| \le b, z = c\right\}.$$

A point P moves along the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad z = c + 1$$

once around, and acts as a source light projecting a shadow of $\mathscr{R}$ onto the xy-plane. Find the area of this shadow.

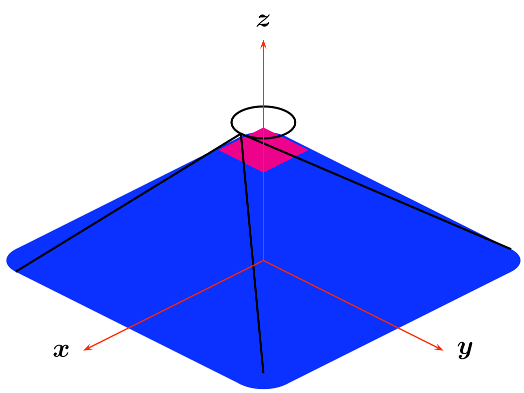

Figure 1.108: Problem 1.16.4.

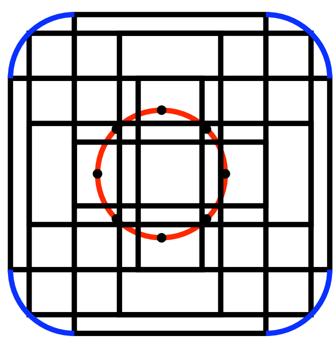

Figure 1.109: Problem 1.16.4.

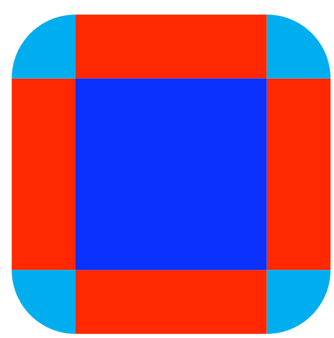

Figure 1.110: Problem 1.16.4.

Solution: ▶ *First consider the same problem as P moves around the circle*

$$x^2 + y^2 = 1, \quad z = c + 1$$

and the region is $\mathcal{R}' = \left\{(x, y, z) \in \mathbb{R}^3 : |x| \le 1, |y| \le 1, z = c\right\}$.

For fixed P $(u, v, c + 1)$ on the circle, the image of $\mathcal{R}'$ (a 2×2 square) on the xy plane is a $(2c + 2) \times (2c + 2)$ square with centre at the point $Q(-cu, -cv, 0)$ (figure 1.109). As P moves along the circle, Q moves along the circle with equation $x^2 + y^2 = c^2$ on the xy-plane (figure 1.109), being the centre of a $(2c+2) \times (2c+2)$ square. This creates a region as in figure 1.110, where each quarter circle has radius c, and the central square has side $2c + 2$, of area

$$\pi c^2 + 4(c + 1)^2 + 8c(c + 1).$$

Resizing to a region

$$\mathcal{R} = \left\{(x, y, z) \in \mathbb{R}^3 : |x| \leq a, |y| \leq b, z = c\right\},$$

and an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = c + 1$$

we use instead of $c+1$, $a(c+1)$ *(parallel to the* x*-axis) and* $b(c+1)$ *(parallel to the* y*-axis), so that the area shadowed is*

$$\pi ab(c+1)^2 + 4ab(c+1)^2 + 4abc(c+1) = c^2ab(\pi + 12) + 16abc + 4ab.$$

◀

Homework

Problem 1.16.1 Let $\mathscr{C}$ be the curve in $\mathbb{R}^3$ defined by

$$x = t^2, \qquad y = 4t^{3/2}, \qquad z = 9t, \qquad t \in [0; +\infty[.$$

Calculate the distance along $\mathscr{C}$ from $(1, 4, 9)$ to $(16, 32, 36)$.

Problem 1.16.2 Consider the surfaces in $\mathbb{R}^3$ implicitly defined by

$$z - x^2 - y^2 - 1 = 0, \qquad z + x^2 + y^2 - 3 = 0.$$

Describe, as vividly as possible these surfaces and their intersection, if they at all intersect. Find a parametric equation for the curve on which they intersect, if they at all intersect.

Problem 1.16.3 Consider the space curve

$$\overrightarrow{r} : t \mapsto \begin{bmatrix} \dfrac{t^4}{1+t^2} \\ \dfrac{t^3}{1+t^2} \\ \dfrac{t^2}{1+t^2} \end{bmatrix}.$$

Let $t_k, 1 \leq k \leq 4$ be non-zero real numbers. Prove that $\overrightarrow{r}(t_1)$, $\overrightarrow{r}(t_2)$, $\overrightarrow{r}(t_3)$, and $\overrightarrow{r}(t_4)$ are coplanar if and only if

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = 0.$$

Problem 1.16.4 Give a parametrisation for the part of the ellipsoid

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

which lies on top of the plane $x + y + z = 0$.

Problem 1.16.5 Let P be the point $(2, 0, 1)$ and consider the curve $\mathscr{C} : z = y^2$ on the yz-plane. As a point Q moves along $\mathscr{C}$, let R be the point of intersection of $\overleftrightarrow{PQ}$ and the xy-plane. Graph all points R on the xy-plane.

Problem 1.16.6 Let a be a real number parameter, and consider the planes

$$P_1 : ax + y + z = -a,$$

$$P_2 : x - ay + az = -1.$$

Let l be their intersection line.

1. Find a direction vector for l.
2. As a varies through $\mathbb{R}$, l describes a surface $\mathcal{S}$ in $\mathbb{R}^3$. Let (x, y, z) be the point of intersection of this surface and the plane $z = c$. Find an equation relating x and y.
3. Find the volume bounded by the two planes, $x = 0$, and $x = 1$, and the surface $\mathcal{S}$ as c varies.

1.17 Multidimensional Vectors

We briefly describe space in n-dimensions. The ideas expounded earlier about the plane and space carry almost without change.

132 Definition $\mathbb{R}^n$ is the n-dimensional space, the collection

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_k \in \mathbb{R} \right\}.$$

133 Definition If $\overrightarrow{a}$ and $\overrightarrow{b}$ are two vectors in $\mathbb{R}^n$ their *vector sum* $\overrightarrow{a}+\overrightarrow{b}$ is defined by the coordinatewise addition

$$\overrightarrow{a}+\overrightarrow{b} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}. \tag{1.20}$$

134 Definition A real number $\alpha \in \mathbb{R}$ will be called a *scalar*. If $\alpha \in \mathbb{R}$ and $\overrightarrow{a} \in \mathbb{R}^n$ we define *scalar multiplication* of a vector and a scalar by the coordinatewise multiplication

$$\alpha\overrightarrow{a} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \end{bmatrix}. \tag{1.21}$$

135 Definition The *standard ordered basis* for $\mathbb{R}^n$ is the collection of vectors

$$\{\overrightarrow{e}_1, \overrightarrow{e}_2, \ldots, \overrightarrow{e}_n, \}$$

with

$$\overrightarrow{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

(a 1 in the k slot and 0's everywhere else). Observe that

$$\sum_{k=1}^{n} \alpha_k \overrightarrow{e}_k = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

136 Definition Given vectors $\overrightarrow{a}$, $\overrightarrow{b}$ of $\mathbb{R}^n$, their *dot product* is

$$\overrightarrow{a}\bullet\overrightarrow{b} = \sum_{k=1}^{n} a_k b_k.$$

We now establish one of the most useful inequalities in analysis.

137 Theorem (Cauchy-Bunyakovsky-Schwarz Inequality) Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be any two vectors in $\mathbb{R}^n$. Then we have

$$|\overrightarrow{x}\bullet\overrightarrow{y}| \leq ||\overrightarrow{x}||||\overrightarrow{y}||.$$

Proof: *Since the norm of any vector is non-negative, we have*

$$\begin{aligned} ||\overrightarrow{x}+t\overrightarrow{y}|| \geq 0 \quad &\iff \quad (\overrightarrow{x}+t\overrightarrow{y})\bullet(\overrightarrow{x}+t\overrightarrow{y}) \geq 0 \\ &\iff \quad \overrightarrow{x}\bullet\overrightarrow{x} + 2t\overrightarrow{x}\bullet\overrightarrow{y} + t^2\overrightarrow{y}\bullet\overrightarrow{y} \geq 0 \\ &\iff \quad ||\overrightarrow{x}||^2 + 2t\overrightarrow{x}\bullet\overrightarrow{y} + t^2||\overrightarrow{y}||^2 \geq 0. \end{aligned}$$

This last expression is a quadratic polynomial in t which is always non-negative. As such its discriminant must be non-positive, that is,

$$(2\overrightarrow{x}\bullet\overrightarrow{y})^2 - 4(\left|\left|\overrightarrow{x}\right|\right|^2)(\left|\left|\overrightarrow{y}\right|\right|^2) \leq 0 \iff |\overrightarrow{x}\bullet\overrightarrow{y}| \leq \left|\left|\overrightarrow{x}\right|\right|\left|\left|\overrightarrow{y}\right|\right|,$$

giving the theorem. ❑

☞ *The above proof works not just for* $\mathbb{R}^n$ *but for any* vector space *(cf. below) that has an inner product.*

The form of the Cauchy-Bunyakovsky-Schwarz most useful to us will be

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} \left(\sum_{k=1}^{n} y_k^2\right)^{1/2}, \tag{1.22}$$

for real numbers x_k, y_k.

138 Corollary (Triangle Inequality) Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be any two vectors in $\mathbb{R}^n$. Then we have

$$\left|\left|\overrightarrow{a} + \overrightarrow{b}\right|\right| \leq \left|\left|\overrightarrow{a}\right|\right| + \left|\left|\overrightarrow{b}\right|\right|.$$

Proof:

$$\begin{aligned}
||\overrightarrow{a} + \overrightarrow{b}||^2 &= (\overrightarrow{a} + \overrightarrow{b})\bullet(\overrightarrow{a} + \overrightarrow{b}) \\
&= \overrightarrow{a}\bullet\overrightarrow{a} + 2\overrightarrow{a}\bullet\overrightarrow{b} + \overrightarrow{b}\bullet\overrightarrow{b} \\
&\leq ||\overrightarrow{a}||^2 + 2||\overrightarrow{a}||||\overrightarrow{b}|| + ||\overrightarrow{b}||^2 \\
&= (||\overrightarrow{a}|| + ||\overrightarrow{b}||)^2,
\end{aligned}$$

from where the desired result follows. ❑

Again, the preceding proof is valid in any vector space that has a norm.

139 Definition Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be two non-zero vectors in a vector space over the real numbers. Then the angle $(\widehat{\overrightarrow{x}, \overrightarrow{y}})$ between them is given by the relation

$$\cos(\widehat{\overrightarrow{x}, \overrightarrow{y}}) = \frac{\overrightarrow{x}\bullet\overrightarrow{y}}{||\overrightarrow{x}||||\overrightarrow{y}||}.$$

This expression agrees with the geometry in the case of the dot product for $\mathbb{R}^2$ and $\mathbb{R}^3$.

140 Example Assume that $a_k, b_k, c_k, k = 1, \ldots, n$, are positive real numbers. Shew that

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leq \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^4\right)\left(\sum_{k=1}^{n} c_k^2\right)^2.$$

Solution: ► *Using CBS on* $\sum_{k=1}^{n}(a_k b_k)c_k$ *once we obtain*

$$\sum_{k=1}^{n} a_k b_k c_k \leq \left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^{1/2} \left(\sum_{k=1}^{n} c_k^2\right)^{1/2}.$$

Using CBS again on $\left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^{1/2}$ *we obtain*

$$\begin{aligned}
\sum_{k=1}^{n} a_k b_k c_k &\leq \left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^{1/2} \left(\sum_{k=1}^{n} c_k^2\right)^{1/2} \\
&\leq \left(\sum_{k=1}^{n} a_k^4\right)^{1/4} \left(\sum_{k=1}^{n} b_k^4\right)^{1/4} \left(\sum_{k=1}^{n} c_k^2\right)^{1/2},
\end{aligned}$$

which gives the required inequality. ◀

We now use the CBS inequality to establish another important inequality. We need some preparatory work.

141 Lemma Let $a_k > 0$, $q_k > 0$, with $\sum_{k=1}^{n} q_k = 1$. Then

$$\lim_{x\to 0}\log\left(\sum_{k=1}^{n} q_k a_k^x\right)^{1/x} = \sum_{k=1}^{n} q_k \log a_k.$$

Proof: *Recall that* $\log(1+x) \sim x$ *as* $x \to 0$. *Thus*

$$\begin{aligned}\lim_{x\to 0}\log\left(\sum_{k=1}^{n} q_k a_k^x\right)^{1/x} &= \lim_{x\to 0}\frac{\log\left(\sum_{k=1}^{n} q_k a_k^x\right)}{x} \\ &= \lim_{x\to 0}\frac{\sum_{k=1}^{n} q_k (a_k^x - 1)}{x} \\ &= \lim_{x\to 0}\sum_{k=1}^{n} q_k \frac{(a_k^x - 1)}{x} \\ &= \sum_{k=1}^{n} q_k \log a_k.\end{aligned}$$

❑

142 Theorem (Arithmetic Mean-Geometric Mean Inequality) Let $a_k \geq 0$. Then

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdot + a_n}{n}.$$

Proof: *If* $b_k \geq 0$, *then by CBS*

$$\frac{1}{n}\sum_{k=1}^{n} b_k \geq \left(\frac{1}{n}\sum_{k=1}^{n}\sqrt{b_k}\right)^2. \tag{1.23}$$

Successive applications of (1.23) yield the monotone decreasing sequence

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \left(\frac{1}{n}\sum_{k=1}^{n}\sqrt{a_k}\right)^2 \geq \left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[4]{a_k}\right)^4 \geq \ldots,$$

which by Lemma 141 has limit

$$\exp\left(\frac{1}{n}\sum_{k=1}^{n}\log a_k\right) = \sqrt[n]{a_1 a_2 \cdots a_n},$$

giving

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdot + a_n}{n},$$

as wanted. ❑

143 Example For any positive integer $n > 1$ we have

$$1\cdot 3\cdot 5\cdots\cdot(2n-1) < n^n.$$

For, by AMGM,

$$1\cdot 3\cdot 5\cdots\cdot(2n-1) < \left(\frac{1+3+5+\cdots+(2n-1)}{n}\right)^n = \left(\frac{n^2}{n}\right)^n = n^n.$$

Notice that since the factors are unequal we have strict inequality.

144 Definition Let $a_1 > 0, a_2 > 0, \ldots, a_n > 0$. Their *harmonic mean* is given by

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

As a corollary to AMGM we obtain

145 Corollary (Harmonic Mean-Geometric Mean Inequality) Let $b_1 > 0, b_2 > 0, \ldots, b_n > 0$. Then

$$\frac{n}{\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}} \leq (b_1 b_2 \cdots b_n)^{1/n}.$$

Proof: *This follows by putting $a_k = \frac{1}{b_k}$ in Theorem 142 . For then*

$$\left(\frac{1}{b_1}\frac{1}{b_2}\cdots\frac{1}{b_n}\right)^{1/n} \leq \frac{\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}}{n}.$$

❑

Combining Theorem 142 and Corollary 145, we deduce

146 Corollary (Harmonic Mean-Arithmetic Mean Inequality) Let $b_1 > 0, b_2 > 0, \ldots, b_n > 0$. Then

$$\frac{n}{\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}} \leq \frac{b_1 + b_2 + \cdots + b_n}{n}.$$

147 Example Let $a_k > 0$, and $s = a_1 + a_2 + \cdots + a_n$. Prove that

$$\sum_{k=1}^{n} \frac{s}{s - a_k} \geq \frac{n^2}{n-1}$$

and

$$\sum_{k=1}^{n} \frac{a_k}{s - a_k} \geq \frac{n}{n-1}.$$

Solution: ► *Put $b_k = \frac{s}{s - a_k}$. Then*

$$\sum_{k=1}^{n} \frac{1}{b_k} = \sum_{k=1}^{n} \frac{s - a_k}{s} = n - 1$$

and from Corollary 146,

$$\frac{n}{n-1} \leq \frac{\sum_{k=1}^{n} \frac{s}{s - a_k}}{n},$$

from where the first inequality is proved.

Since $\dfrac{s}{s-a_k} - 1 = \dfrac{a_k}{s-a_k}$, *we have*

$$\begin{aligned}\sum_{k=1}^{n} \frac{a_k}{s-a_k} &= \sum_{k=1}^{n}\left(\frac{s}{s-a_k} - 1\right) \\ &= \sum_{k=1}^{n}\left(\frac{s}{s-a_k}\right) - n \\ &\geq \frac{n^2}{n-1} - n \\ &= \frac{n}{n-1}.\end{aligned}$$

◀

Homework

Problem 1.17.1 The *Arithmetic Mean Geometric Mean Inequality* says that if $a_k \geq 0$ then

$$(a_1a_2\cdots a_n)^{1/n} \leq \frac{a_1+a_2+\cdots+a_n}{n}.$$

Equality occurs if and only if $a_1 = a_2 = \ldots = a_n$. In this exercise you will follow the steps of a proof by George Pólya.

1. Prove that $\forall x \in \mathbb{R},\ x \leq e^{x-1}$.
2. Put
$$A_k = \frac{na_k}{a_1+a_2+\cdots+a_n},$$
and $G_n = a_1a_2\cdots a_n$. Prove that
$$A_1A_2\cdots A_n = \frac{n^nG_n}{(a_1+a_2+\cdots+a_n)^n},$$
and that
$$A_1+A_2+\cdots+A_n = n.$$
3. Deduce that
$$G_n \leq \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n.$$
4. Prove the AMGM inequality by assembling the results above.

Problem 1.17.2 Demonstrate that if $x_1, x_2, \ldots, x_n$, are strictly positive real numbers then

$$(x_1+x_2+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right) \geq n^2.$$

Problem 1.17.3 (USAMO 1978) Let a, b, c, d, e be real numbers such that

$$a+b+c+d+e = 8, \qquad a^2+b^2+c^2+d^2+e^2 = 16.$$

Maximise the value of e.

Problem 1.17.4 Find all positive real numbers

$$a_1 \leq a_2 \leq \ldots \leq a_n$$

such that

$$\sum_{k=1}^{n} a_k = 96, \qquad \sum_{k=1}^{n} a_k^2 = 144, \qquad \sum_{k=1}^{n} a_k^3 = 216.$$

Problem 1.17.5 Demonstrate that for integer $n > 1$ we have,

$$n! < \left(\frac{n+1}{2}\right)^n.$$

Problem 1.17.6 Let $f(x) = (a+x)^5(a-x)^3$, $x \in [-a; a]$. Find the maximum value of de f using the AM-GM inequality.

Problem 1.17.7 Prove that the sequence $x_n = \left(1+\dfrac{1}{n}\right)^n$, $n = 1, 2, \ldots$ is strictly increasing.

2 Differentiation

2.1 Some Topology

148 Definition Let $\mathbf{a} \in \mathbb{R}^n$ and let $\varepsilon > 0$. An *open ball* centred at $\mathbf{a}$ of radius ε is the set

$$B_\varepsilon(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < \varepsilon\}.$$

An *open box* is a Cartesian product of open intervals

$$]a_1; b_1[\times]a_2; b_2[\times \cdots \times]a_{n-1}; b_{n-1}[\times]a_n; b_n[,$$

where the a_k, b_k are real numbers.

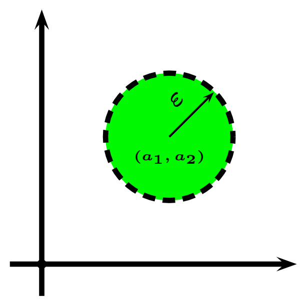

Figure 2.1: Open ball in $\mathbb{R}^2$.

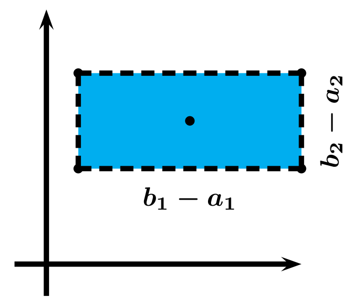

Figure 2.2: Open rectangle in $\mathbb{R}^2$.

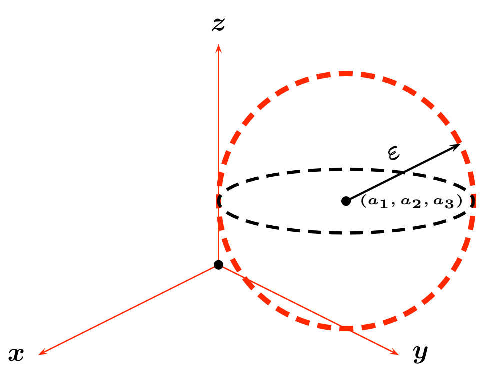

Figure 2.3: Open ball in $\mathbb{R}^3$.

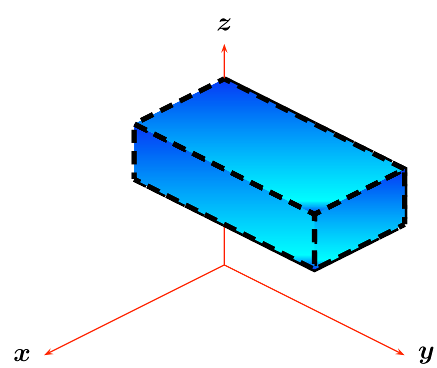

Figure 2.4: Open box in $\mathbb{R}^3$.

149 Example An open ball in $\mathbb{R}$ is an open interval, an open ball in $\mathbb{R}^2$ is an open disk (see figure 2.1) and an open ball in $\mathbb{R}^3$ is an open sphere (see figure 2.3). An open box in $\mathbb{R}$ is an open interval, an open box in $\mathbb{R}^2$ is a rectangle without its boundary (see figure 2.2) and an open box in $\mathbb{R}^3$ is a box without its boundary (see figure 2.4).

150 Definition A set $\mathscr{O} \subseteq \mathbb{R}^n$ is said to be *open* if for every point belonging to it we can surround the point by a sufficiently small open ball so that this balls lies completely within the set. That is, $\forall \mathrm{a} \in \mathscr{O} \ \ \exists \varepsilon > 0$ such that $B_\varepsilon(\mathrm{a}) \subseteq \mathscr{O}$.

151 Example The open interval $]-1;1[$ is open in $\mathbb{R}$. The interval $]-1;1]$ is not open, however, as no interval centred at 1 is totally contained in $]-1;1]$.

152 Example The region $]-1;1[\times]0;+\infty[$ is open in $\mathbb{R}^2$.

153 Example The ellipsoidal region $\left\{(x,y) \in \mathbb{R}^2 : x^2 + 4y^2 < 4\right\}$ is open in $\mathbb{R}^2$.

The reader will recognise that open boxes, open ellipsoids and their unions and finite intersections are open sets in $\mathbb{R}^n$.

154 Definition A set $\mathscr{F} \subseteq \mathbb{R}^n$ is said to be *closed* in $\mathbb{R}^n$ if its complement $\mathbb{R}^n \setminus \mathscr{F}$ is open.

155 Example The closed interval $[-1;1]$ is closed in $\mathbb{R}$, as its complement, $\mathbb{R}\setminus[-1;1] =]-\infty;-1[\cup]1;+\infty[$ is open in $\mathbb{R}$. The interval $]-1;1]$ is neither open nor closed in $\mathbb{R}$, however.

156 Example The region $[-1;1] \times [0;+\infty[\times[0;2]$ is closed in $\mathbb{R}^3$.

Homework

Problem 2.1.1 Determine whether the following subsets of $\mathbb{R}^2$ are open, closed, or neither, in $\mathbb{R}^2$.

1. $A = \{(x,y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}$
2. $B = \{(x,y) \in \mathbb{R}^2 : |x| < 1, |y| \leq 1\}$
3. $C = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$
4. $D = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq x\}$
5. $E = \{(x,y) \in \mathbb{R}^2 : xy > 1\}$
6. $F = \{(x,y) \in \mathbb{R}^2 : xy \leq 1\}$
7. $G = \{(x,y) \in \mathbb{R}^2 : |y| \leq 9, x < y^2\}$

Problem 2.1.2 (Putnam Exam 1969) Let $p(x,y)$ be a polynomial with real coefficients in the real variables x and y, defined over the entire plane $\mathbb{R}^2$. What are the possibilities for the image (range) of $p(x,y)$?

Problem 2.1.3 (Putnam 1998) Let $\mathcal{F}$ be a finite collection of open disks in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Shew that there is a pairwise disjoint subcollection $D_k, k \geq 1$ in $\mathcal{F}$ such that

$$E \subseteq \bigcup_{j=1}^{n} 3D_j.$$

2.2 Multivariable Functions

Let $A \subseteq \mathbb{R}^n$. For most of this course, our concern will be functions of the form

$$f : A \to \mathbb{R}^m.$$

If $m = 1$, we say that f is a *scalar field*. If $m \geq 2$, we say that f is a *vector field*.

We would like to develop a calculus analogous to the situation in $\mathbb{R}$. In particular, we would like to examine limits, continuity, differentiability, and integrability of multivariable functions. Needless to say, the introduction of more variables greatly complicates the analysis. For example, recall that the graph of a function $f : A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$. is the set

$$\{(\mathrm{x}, f(\mathrm{x})) : \mathrm{x} \in A)\} \subseteq \mathbb{R}^{n+m}.$$

If $m + n > 3$, we have an object of more than three-dimensions! In the case $n = 2, m = 1$, we have a tri-dimensional surface. We will now briefly examine this case.

157 Definition Let $A \subseteq \mathbb{R}^2$ and let $f : A \to \mathbb{R}$ be a function. Given $c \in \mathbb{R}$, the *level curve* at $z = c$ is the curve resulting from the intersection of the surface $z = f(x, y)$ and the plane $z = c$, if there is such a curve.

158 Example The level curves of the surface $f(x, y) = x^2 + y^2$ (an elliptic paraboloid) are the concentric circles

$$x^2 + y^2 = c, \qquad c > 0.$$

Homework

Problem 2.2.1 Sketch the level curves for the following maps.

1. $(x, y) \mapsto x + y$
2. $(x, y) \mapsto xy$
3. $(x, y) \mapsto \min(|x|, |y|)$
4. $(x, y) \mapsto x^3 - x$
5. $(x, y) \mapsto x^2 + 4y^2$
6. $(x, y) \mapsto \sin(x^2 + y^2)$
7. $(x, y) \mapsto \cos(x^2 - y^2)$

Problem 2.2.2 Sketch the level surfaces for the following maps.

1. $(x, y, z) \mapsto x + y + z$
2. $(x, y, z) \mapsto xyz$
3. $(x, y, z) \mapsto \min(|x|, |y|, |z|)$
4. $(x, y, z) \mapsto x^2 + y^2$
5. $(x, y, z) \mapsto x^2 + 4y^2$
6. $(x, y, z) \mapsto \sin(z - x^2 - y^2)$
7. $(x, y, z) \mapsto x^2 + y^2 + z^2$

2.3 Limits

We will start with the notion of *limit*.

159 Definition A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to have a limit $\mathbf{L} \in \mathbb{R}^m$ at $\mathbf{a} \in \mathbb{R}^n$ if $\forall \epsilon > 0 \exists \delta > 0$ such that

$$0 < ||\mathbf{x} - \mathbf{a}|| < \delta \implies ||f(\mathbf{x}) - \mathbf{L}|| < \epsilon.$$

In such a case we write,

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{L}.$$

The notions of infinite limits, limits at infinity, and continuity at a point, are analogously defined. Limits in more than one dimension are perhaps trickier to find, as one must approach the test point from infinitely many directions.

160 Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$.

Solution: ► *We use the sandwich theorem. Observe that* $0 \leq x^2 \leq x^2 + y^2$, *and so* $0 \leq \dfrac{x^2}{x^2 + y^2} \leq 1$. *Thus*

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \left|\frac{x^2 y}{x^2 + y^2}\right| \leq \lim_{(x,y)\to(0,0)} |y|,$$

and hence

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0.$$

The Maple™ commands to graph this surface and find this limits appear below. Notice that Maple is unable to find the limit and so returns unevaluated.

```
> with(plots):
> plot3d(x^2*y/(x^2+y^2),x=-10..10,y=-10..10,axes=boxed,color=x^2+y^2);
> limit(x^2*y/(x^2+y^2),x=0,y=0);
```

◀

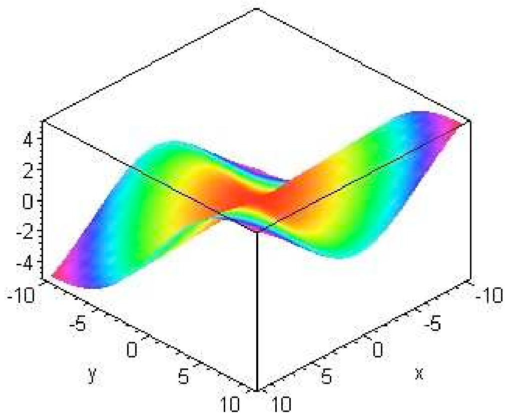

Figure 2.5: $(x, y) \mapsto \dfrac{x^2 y}{x^2 + y^2}$.

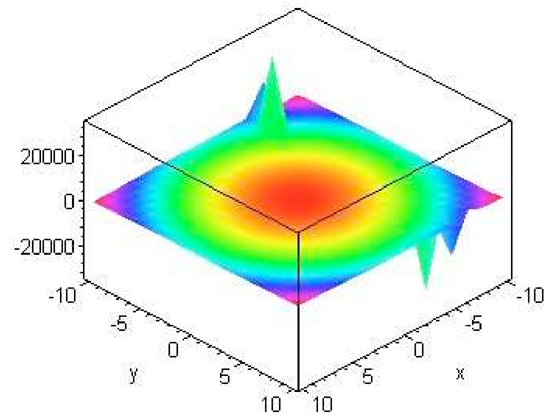

Figure 2.6: $(x, y) \mapsto \dfrac{x^5 y^3}{x^6 + y^4}$.

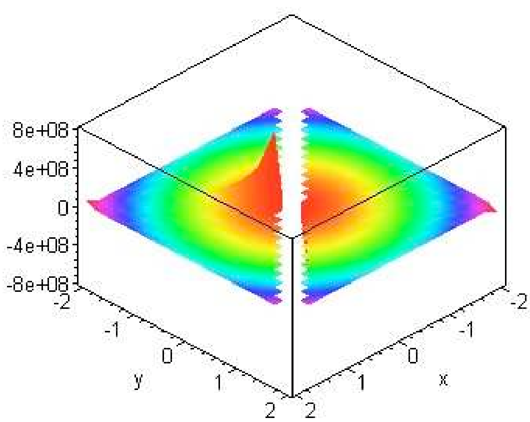

Figure 2.7: Example 162.

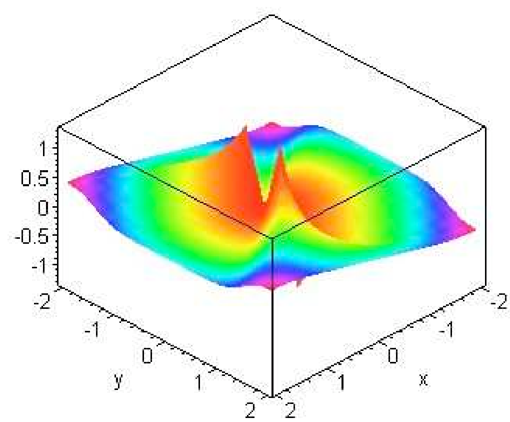

Figure 2.8: Example 163.

161 Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^5 y^3}{x^6 + y^4}$.

Solution: ► *Either* $|x| \leq |y|$ *or* $|x| \geq |y|$. *Observe that if* $|x| \leq |y|$, *then*

$$\left|\frac{x^5y^3}{x^6+y^4}\right| \leq \frac{y^8}{y^4} = y^4.$$

If $|y| \leq |x|$, *then*

$$\left|\frac{x^5y^3}{x^6+y^4}\right| \leq \frac{x^8}{x^6} = x^2.$$

Thus

$$\left|\frac{x^5y^3}{x^6+y^4}\right| \leq \max(y^4, x^2) \leq y^4 + x^2 \longrightarrow 0,$$

as $(x, y) \to (0, 0)$.
Aliter: *Let* $X = x^3, Y = y^2$.

$$\left|\frac{x^5y^3}{x^6+y^4}\right| = \frac{X^{5/3}Y^{3/2}}{X^2+Y^2}.$$

Passing to polar coordinates $X = \rho\cos\theta, Y = \rho\sin\theta$, *we obtain*

$$\left|\frac{x^5y^3}{x^6+y^4}\right| = \frac{X^{5/3}Y^{3/2}}{X^2+Y^2} = \rho^{5/3+3/2-2}|\cos\theta|^{5/3}|\sin\theta|^{3/2} \leq \rho^{7/6} \to 0,$$

as $(x, y) \to (0, 0)$. ◄

162 Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1+x+y}{x^2-y^2}$.

Solution: ► *When* $y = 0$,

$$\frac{1+x}{x^2} \to +\infty,$$

as $x \to 0$. *When* $x = 0$,

$$\frac{1+y}{-y^2} \to -\infty,$$

as $y \to 0$. *The limit does not exist.* ◄

163 Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^6}{x^6+y^8}$.

Solution: ► *Putting* $x = t^4, y = t^3$, *we find*

$$\frac{xy^6}{x^6+y^8} = \frac{1}{2t^2} \to +\infty,$$

as $t \to 0$. *But when* $y = 0$, *the function is* 0. *Thus the limit does not exist.* ◄

164 Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{((x-1)^2+y^2)\log_e((x-1)^2+y^2)}{|x|+|y|}$.

Solution: ► *When* $y = 0$ *we have*

$$\frac{2(x-1)^2\ln(|1-x|)}{|x|} \sim -\frac{2x}{|x|},$$

and so the function does not have a limit at $(0, 0)$. ◄

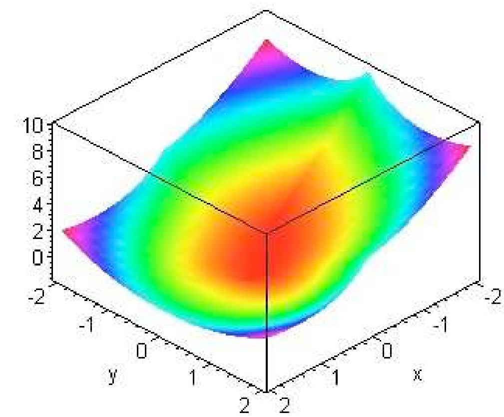

Figure 2.9: Example 164.

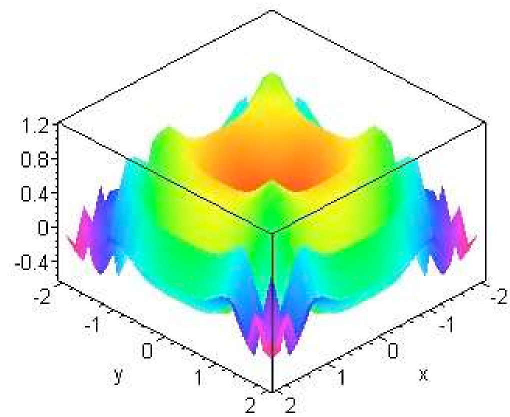

Figure 2.10: Example 165.

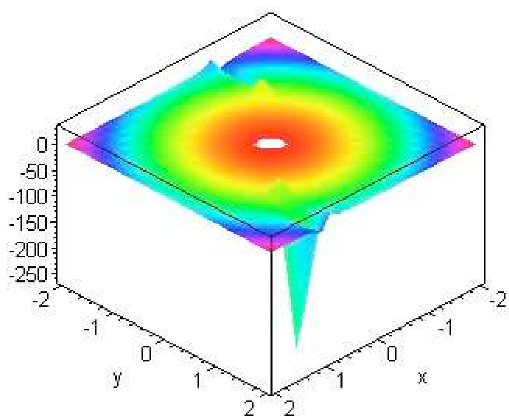

Figure 2.11: Example 166.

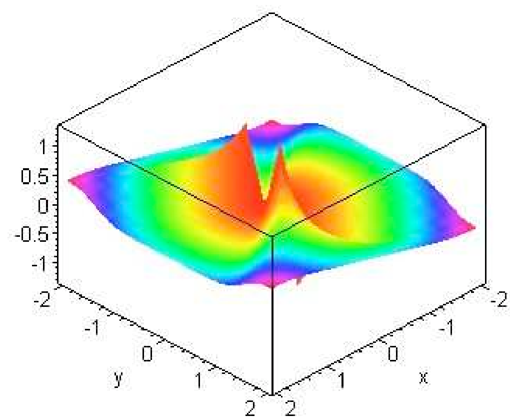

Figure 2.12: Example 163.

165 Example Find $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^4)+\sin(y^4)}{\sqrt{x^4+y^4}}$.

Solution: ► *$\sin(x^4)+\sin(y^4) \leq x^4+y^4$ and so*

$$\left|\frac{\sin(x^4)+\sin(y^4)}{\sqrt{x^4+y^4}}\right| \leq \sqrt{x^4+y^4} \to 0,$$

as $(x,y) \to (0,0)$. ◄

166 Example Find $\lim_{(x,y)\to(0,0)} \dfrac{\sin x - y}{x - \sin y}$.

Solution: ► *When $y = 0$ we obtain*

$$\frac{\sin x}{x} \to 1,$$

as $x \to 0$. When $y = x$ the function is identically -1. Thus the limit does not exist. ◄

If $f : \mathbb{R}^2 \to \mathbb{R}$, it may be that the limits

$$\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right), \qquad \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right),$$

both exist. These are called the *iterated limits of f as* $(x,y) \to (x_0,y_0)$. The following possibilities might occur.

1. If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, then each of the iterated limits $\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right)$ and $\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right)$ exists.
2. If the iterated limits exist and $\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right) \neq \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right)$ then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.
3. It may occur that $\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right) = \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right)$, but that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.
4. It may occur that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, but one of the iterated limits does not.

Homework

Problem 2.3.1 Sketch the domain of definition of $(x,y) \mapsto \sqrt{4-x^2-y^2}$.

Problem 2.3.2 Sketch the domain of definition of $(x,y) \mapsto \log(x+y)$.

Problem 2.3.3 Sketch the domain of definition of $(x,y) \mapsto \dfrac{1}{x^2+y^2}$.

Problem 2.3.4 Find $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\dfrac{1}{xy}$.

Problem 2.3.5 Find $\lim_{(x,y)\to(0,2)} \dfrac{\sin xy}{x}$.

Problem 2.3.6 For what c will the function

$$f(x,y) = \begin{cases} \sqrt{1-x^2-4y^2}, & \text{if } x^2+4y^2 \leq 1, \\ c, & \text{if } x^2+4y^2 > 1 \end{cases}$$

be continuous everywhere on the xy-plane?

Problem 2.3.7 Find

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\sin\frac{1}{x^2+y^2}.$$

Problem 2.3.8 Find

$$\lim_{(x,y)\to(+\infty,+\infty)} \frac{\max(|x|,|y|)}{\sqrt{x^4+y^4}}.$$

Problem 2.3.9 Find

$$\lim_{(x,y)\to(0,0} \frac{2x^2\sin y^2+y^4e^{-|x|}}{\sqrt{x^2+y^2}}.$$

Problem 2.3.10 Demonstrate that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2} = 0.$$

Problem 2.3.11 Prove that

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{x+y}\right) = 1 = -\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{x+y}\right).$$

Does $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x-y}{x+y}$ exist?.

Problem 2.3.12 Let

$$f(x,y) = \begin{cases} x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y} & \text{if } x\neq 0,\ y\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists, but that the iterated limits $\displaystyle\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ and $\displaystyle\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$ do not exist.

Problem 2.3.13 Prove that

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right) = 0,$$

and that

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right) = 0,$$

but still $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}$ does not exist.

2.4 Definition of the Derivative

Before we begin, let us introduce some necessary notation. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. We write $f(h) = \mathrm{o}(h)$ if $f(h)$ goes faster to 0 than h, that is, if $\displaystyle\lim_{h\to 0}\frac{f(h)}{h} = 0$. For example, $h^3+2h^2 = \mathrm{o}(h)$, since

$$\lim_{h\to 0}\frac{h^3+2h^2}{h} = \lim_{h\to 0} h^2+2h = 0.$$

We now define the derivative in the multidimensional space $\mathbb{R}^n$. Recall that in one variable, a function $g:\mathbb{R}\to\mathbb{R}$ is said to be differentiable at $x=a$ if the limit

$$\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = g'(a)$$

exists. The limit condition above is equivalent to saying that

$$\lim_{x\to a}\frac{g(x)-g(a)-g'(a)(x-a)}{x-a} = 0,$$

or equivalently,

$$\lim_{h\to 0}\frac{g(a+h)-g(a)-g'(a)(h)}{h} = 0.$$

We may write this as

$$g(a+h)-g(a) = g'(a)(h)+\mathrm{o}(h).$$

The above analysis provides an analogue definition for the higher-dimensional case. Observe that since we may not divide by vectors, the corresponding definition in higher dimensions involves quotients of norms.

167 Definition Let $A\subseteq\mathbb{R}^n$. A function $f:A\to\mathbb{R}^m$ is said to be *differentiable* at $\mathrm{a}\in A$ if there is a linear transformation, called the *derivative of f at* a, $\mathscr{D}_{\mathrm{a}}(f):\mathbb{R}^n\to\mathbb{R}^m$ such that

$$\lim_{\mathrm{x}\to\mathrm{a}}\frac{||f(\mathrm{x})-f(\mathrm{a})-\mathscr{D}_{\mathrm{a}}(f)(\mathrm{x}-\mathrm{a})||}{||\mathrm{x}-\mathrm{a}||} = 0.$$

Equivalently, f is differentiable at $\mathbf{a}$ if there is a linear transformation $\mathscr{D}_{\mathbf{a}}(f)$ such that

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = \mathscr{D}_{\mathbf{a}}(f)(\mathbf{h}) + o\left(||\mathbf{h}||\right),$$

as $\mathbf{h} \to \mathbf{0}$.

The condition for differentiability at $\mathbf{a}$ is equivalent to

$$f(\mathbf{x}) - f(\mathbf{a}) = \mathscr{D}_{\mathbf{a}}(f)(\mathbf{x}-\mathbf{a}) + o\left(||\mathbf{x}-\mathbf{a}||\right),$$

as $\mathbf{x} \to \mathbf{a}$.

168 Theorem If A is an open set in definition 167, $\mathscr{D}_{\mathbf{a}}(f)$ is uniquely determined.

Proof: *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be another linear transformation satisfying definition 167. We must prove that $\forall \mathbf{v} \in \mathbb{R}^n, L(\mathbf{v}) = \mathscr{D}_{\mathbf{a}}(f)(\mathbf{v})$. Since A is open, $\mathbf{a}+\mathbf{h} \in A$ for sufficiently small $||\mathbf{h}||$. By definition, as $\mathbf{h} \to \mathbf{0}$, we have*

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = \mathscr{D}_{\mathbf{a}}(f)(\mathbf{h}) + o\left(||\mathbf{h}||\right).$$

and

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = L(\mathbf{h}) + o\left(||\mathbf{h}||\right).$$

Now, observe that

$$\mathscr{D}_{\mathbf{a}}(f)(\mathbf{v}) - L(\mathbf{v}) = \mathscr{D}_{\mathbf{a}}(f)(\mathbf{h}) - f(\mathbf{a}+\mathbf{h}) + f(\mathbf{a}) + f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h}).$$

By the triangle inequality,

$$\begin{array}{lll} ||\mathscr{D}_{\mathbf{a}}(f)(\mathbf{v}) - L(\mathbf{v})|| & \leq & ||\mathscr{D}_{\mathbf{a}}(f)(\mathbf{h}) - f(\mathbf{a}+\mathbf{h}) + f(\mathbf{a})|| \\ & & \quad +||f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})|| \\ & = & o\left(||\mathbf{h}||\right) + o\left(||\mathbf{h}||\right) \\ & = & o\left(||\mathbf{h}||\right), \end{array}$$

as $\mathbf{h} \to \mathbf{0}$ This means that

$$||L(\mathbf{v}) - \mathscr{D}_{\mathbf{a}}(f)(\mathbf{v})|| \to 0,$$

i.e., $L(\mathbf{v}) = \mathscr{D}_{\mathbf{a}}(f)(\mathbf{v})$, completing the proof. ❑

If $A = \{\mathbf{a}\}$, a singleton, then $\mathscr{D}_{\mathbf{a}}(f)$ is not uniquely determined. For $||\mathbf{x}-\mathbf{a}|| < \delta$ holds only for $\mathbf{x} = \mathbf{a}$, and so $f(\mathbf{x}) = f(\mathbf{a})$. Any linear transformation T will satisfy the definition, as $T(\mathbf{x}-\mathbf{a}) = T(\mathbf{0}) = \mathbf{0}$, and

$$||f(\mathbf{x}) - f(\mathbf{a}) - T(\mathbf{x}-\mathbf{a})|| = ||\mathbf{0}|| = 0,$$

identically.

169 Example If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\mathscr{D}_{\mathbf{a}}(L) = L$, for any $\mathbf{a} \in \mathbb{R}^n$.

Solution: ► *Since $\mathbb{R}^n$ is an open set, we know that $\mathscr{D}_{\mathbf{a}}(L)$ uniquely determined. Thus if L satisfies definition 167, then the claim is established. But by linearity*

$$||L(\mathbf{x}) - L(\mathbf{a}) - L(\mathbf{x}-\mathbf{a})|| = ||L(\mathbf{x}) - L(\mathbf{a}) - L(\mathbf{x}) + L(\mathbf{a})|| = ||\mathbf{0}|| = 0,$$

whence the claim follows. ◄

170 Example Let

$$f : \begin{array}{ccc} \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R} \\ (\overrightarrow{x}, \overrightarrow{y}) & \mapsto & \overrightarrow{x}\bullet\overrightarrow{y} \end{array}$$

be the usual dot product in $\mathbb{R}^3$. Shew that f is differentiable and that

$$\mathscr{D}_{(\overrightarrow{x}, \overrightarrow{y})} f(\overrightarrow{h}, \overrightarrow{k}) = \overrightarrow{x}\bullet\overrightarrow{k} + \overrightarrow{h}\bullet\overrightarrow{y}.$$

Solution: ► *We have*

$$\begin{aligned} f(\overrightarrow{x}+\overrightarrow{h},\overrightarrow{y}+\overrightarrow{k})-f(\overrightarrow{x},\overrightarrow{y}) &= (\overrightarrow{x}+\overrightarrow{h})\bullet(\overrightarrow{y}+\overrightarrow{k})-\overrightarrow{x}\bullet\overrightarrow{y} \\ &= \overrightarrow{x}\bullet\overrightarrow{y}+\overrightarrow{x}\bullet\overrightarrow{k}+\overrightarrow{h}\bullet\overrightarrow{y}+\overrightarrow{h}\bullet\overrightarrow{k}-\overrightarrow{x}\bullet\overrightarrow{y} \\ &= \overrightarrow{x}\bullet\overrightarrow{k}+\overrightarrow{h}\bullet\overrightarrow{y}+\overrightarrow{h}\bullet\overrightarrow{k}. \end{aligned}$$

As $(\overrightarrow{h},\overrightarrow{k})\to(\overrightarrow{0},\overrightarrow{0})$, *we have by the Cauchy-Buniakovskii-Schwarz inequality,* $|\overrightarrow{h}\bullet\overrightarrow{k}|\leq\left|\left|\overrightarrow{h}\right|\right|\left|\left|\overrightarrow{k}\right|\right|=o\left(\left|\left|\overrightarrow{h}\right|\right|\right)$, *which proves the assertion.* ◄

Just like in the one variable case, differentiability at a point, implies continuity at that point.

171 Theorem Suppose $A\subseteq\mathbb{R}^n$ is open and $f:A\to\mathbb{R}^n$ is differentiable on A. Then f is continuous on A.

Proof: *Given* $\mathrm{a}\in A$, *we must shew that*

$$\lim_{\mathrm{x}\to\mathrm{a}} f(\mathrm{x})=f(\mathrm{a}).$$

Since f *is differentiable at* a *we have*

$$f(\mathrm{x})-f(\mathrm{a})=\mathscr{D}_{\mathrm{a}}(f)(\mathrm{x}-\mathrm{a})+o\left(||\mathrm{x}-\mathrm{a}||\right),$$

and so

$$f(\mathrm{x})-f(\mathrm{a})\to 0,$$

as $\mathrm{x}\to\mathrm{a}$, *proving the theorem.* ❑

Homework

Problem 2.4.1 Let $L:\mathbb{R}^3\to\mathbb{R}^3$ be a linear transformation and

$$F:\begin{array}{ccc}\mathbb{R}^3 & \to & \mathbb{R}^3 \\ \overrightarrow{x} & \mapsto & \overrightarrow{x}\times L(\overrightarrow{x})\end{array}.$$

Shew that F is differentiable and that

$$\mathscr{D}_{\mathrm{x}}(F)(\overrightarrow{h})=\overrightarrow{x}\times L(\overrightarrow{h})+\overrightarrow{h}\times L(\overrightarrow{x}).$$

Problem 2.4.2 Let $f:\mathbb{R}^n\to\mathbb{R}, n\geq 1, f(\overrightarrow{x})=\left|\left|\overrightarrow{x}\right|\right|$ be the usual norm in $\mathbb{R}^n$, with $\left|\left|\overrightarrow{x}\right|\right|^2=\overrightarrow{x}\bullet\overrightarrow{x}$. Prove that

$$\mathscr{D}_{\mathrm{x}}(f)(\overrightarrow{v})=\frac{\overrightarrow{x}\bullet\overrightarrow{v}}{\left|\left|\overrightarrow{x}\right|\right|},$$

for $\overrightarrow{x}\neq\overrightarrow{0}$, but that f is not differentiable at $\overrightarrow{0}$.

2.5 The Jacobi Matrix

We now establish a way which simplifies the process of finding the derivative of a function at a given point.

172 Definition Let $A\subseteq\mathbb{R}^n$, $f:A\to\mathbb{R}^m$, and put

$$f(\mathrm{x})=\begin{bmatrix} f_1(x_1,x_2,\ldots,x_n) \\ f_2(x_1,x_2,\ldots,x_n) \\ \vdots \\ f_m(x_1,x_2,\ldots,x_n)\end{bmatrix}.$$

Here $f_i:\mathbb{R}^n\to\mathbb{R}$. The *partial derivative* $\dfrac{\partial f_i}{\partial x_j}(\mathrm{x})$ is defined as

$$\frac{\partial f_i}{\partial x_j}(\mathrm{x})=\lim_{h\to 0}\frac{f_i(x_1,x_2,\ldots,x_j+h,\ldots,x_n)-f_i(x_1,x_2,\ldots,x_j,\ldots,x_n)}{h},$$

whenever this limit exists.

To find partial derivatives with respect to the j-th variable, we simply keep the other variables fixed and differentiate with respect to the j-th variable.

173 Example If $f : \mathbb{R}^3 \to \mathbb{R}$, and $f(x, y, z) = x + y^2 + z^3 + 3xy^2z^3$ then

$$\frac{\partial f}{\partial x}(x, y, z) = 1 + 3y^2z^3,$$

$$\frac{\partial f}{\partial y}(x, y, z) = 2y + 6xyz^3,$$

and

$$\frac{\partial f}{\partial z}(x, y, z) = 3z^2 + 9xy^2z^2.$$

The Maple™ commands to find these follow.

```
>  f:=(x,y,z)->x+y^2+z^3+3*x*y^2*z^3;
>  diff(f(x,y,z),x);
>  diff(f(x,y,z),y);
>  diff(f(x,y,z),z);
```

Since the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, it can be represented by aid of matrices. The following theorem will allow us to determine the matrix representation for $\mathscr{D}_{\mathrm{a}}(f)$ under the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

174 Theorem Let

$$f(\mathrm{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}.$$

Suppose $A \subseteq \mathbb{R}^n$ is an open set and $f : A \to \mathbb{R}^m$ is differentiable. Then each partial derivative $\dfrac{\partial f_i}{\partial x_j}(\mathrm{x})$ exists, and the matrix representation of $\mathscr{D}_{\mathrm{x}}(f)$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is the *Jacobi matrix*

$$f'(\mathrm{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathrm{x}) & \frac{\partial f_1}{\partial x_2}(\mathrm{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathrm{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathrm{x}) & \frac{\partial f_2}{\partial x_2}(\mathrm{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathrm{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathrm{x}) & \frac{\partial f_n}{\partial x_2}(\mathrm{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathrm{x}) \end{bmatrix}.$$

Proof: *Let $\overrightarrow{\mathrm{e}}_j, 1 \leq j \leq n$, be the standard basis for $\mathbb{R}^n$. To obtain the Jacobi matrix, we must compute $\mathscr{D}_{\mathrm{x}}(f)(\overrightarrow{\mathrm{e}}_j)$, which will give us the j-th column of the Jacobi matrix. Let $f'(\mathrm{x}) = (J_{ij})$, and observe that*

$$\mathscr{D}_{\mathrm{x}}(f)(\overrightarrow{\mathrm{e}}_j) = \begin{bmatrix} J_{1j} \\ J_{2j} \\ \vdots \\ J_{nj} \end{bmatrix}.$$

and put $\mathrm{y} = \mathrm{x} + \varepsilon\overrightarrow{\mathrm{e}}_j, \varepsilon \in \mathbb{R}$. Notice that

$$\frac{||f(\mathrm{y}) - f(\mathrm{x}) - \mathscr{D}_{\mathrm{x}}(f)(\mathrm{y} - \mathrm{x})||}{||\mathrm{y} - \mathrm{x}||}$$
$$= \frac{||f(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_j, \ldots, x_n) - \varepsilon\mathscr{D}_{\mathrm{x}}(f)(\overrightarrow{\mathrm{e}}_j)||}{|\varepsilon|}.$$

Since the sinistral side $\to 0$ as $\varepsilon \to 0$, the so does the i-th component of the numerator, and so,

$$\frac{|f_i(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f_i(x_1, x_2, \ldots, x_j, \ldots, x_n) - \varepsilon J_{ij}|}{|\varepsilon|} \to 0.$$

This entails that

$$J_{ij} = \lim_{\varepsilon \to 0} \frac{f_i(x_1, x_2, \ldots, x_j + \varepsilon, \ldots, x_n) - f_i(x_1, x_2, \ldots, x_j, \ldots, x_n)}{\varepsilon} = \frac{\partial f_i}{\partial x_j}(\mathrm{x}).$$

This finishes the proof. ❑

☞ *Strictly speaking, the Jacobi matrix is not the derivative of a function at a point. It is a matrix representation of the derivative in the standard basis of $\mathbb{R}^n$. We will abuse language, however, and refer to f' when we mean the Jacobi matrix of f.*

175 Example Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x, y) = (xy + yz, \log_e xy).$$

Compute the Jacobi matrix of f.

Solution: ▶ *The Jacobi matrix is the 2×3 matrix*

$$f'(x, y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x}(x, y) & \dfrac{\partial f_1}{\partial y}(x, y) & \dfrac{\partial f_1}{\partial z}(x, y) \\ \dfrac{\partial f_2}{\partial x}(x, y) & \dfrac{\partial f_2}{\partial y}(x, y) & \dfrac{\partial f_2}{\partial z}(x, y) \end{bmatrix} = \begin{bmatrix} y & x + z & y \\ \dfrac{1}{x} & \dfrac{1}{y} & 0 \end{bmatrix}.$$

◀

176 Example Let $f(\rho, \theta, z) = (\rho \cos \theta, \rho \sin \theta, z)$ be the function which changes from cylindrical coordinates to Cartesian coordinates. We have

$$f'(\rho, \theta, z) = \begin{bmatrix} \cos \theta & -\rho \sin \theta & 0 \\ \sin \theta & \rho \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

177 Example Let $f(\rho, \phi, \theta) = (\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi)$ be the function which changes from spherical coordinates to Cartesian coordinates. We have

$$f'(\rho, \phi, \theta) = \begin{bmatrix} \cos \theta \sin \phi & \rho \cos \theta \cos \phi & -\rho \sin \phi \sin \theta \\ \sin \theta \sin \phi & \rho \sin \theta \cos \phi & \rho \cos \theta \sin \phi \\ \cos \phi & -\rho \sin \phi & 0 \end{bmatrix}.$$

The Jacobi matrix provides a convenient computational tool to compute the derivative of a function at a point. Thus differentiability at a point implies that the partial derivatives of the function exist at the point. The converse, however, is not true.

178 Example Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} y & \text{if} \quad x = 0, \\ x & \text{if} \quad y = 0, \\ 1 & \text{if} \quad xy \neq 0. \end{cases}$$

Observe that f is not continuous at $(0,0)$ ($f(0,0) = 0$ but $f(x,y) = 1$ for values arbitrarily close to $(0,0)$), and hence, it is not differentiable there. We have however, $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 1$. Thus even if both partial derivatives exist at $(0,0)$ is no guarantee that the function will be differentiable at $(0,0)$. You should also notice that both partial derivatives are not continuous at $(0,0)$.

We have, however, the following.

179 Theorem Let $A \subseteq \mathbb{R}^n$ be an open set, and let $f : \mathbb{R}^n \to \mathbb{R}^m$. Put $f = \begin{bmatrix} f_1 \\ f_2 \\ \cdots \\ f_m \end{bmatrix}$. If each of the partial derivatives $\mathscr{D}_j f_i$ exists and is continuous on A, then f is differentiable on A.

The concept of *repeated partial derivatives* is akin to the concept of repeated differentiation. Similarly with the concept of implicit partial differentiation. The following examples should be self-explanatory.

180 Example Let $f(u,v,w) = e^u v \cos w$. Determine $\dfrac{\partial^2}{\partial u \partial v} f(u,v,w)$ at $(1,-1,\dfrac{\pi}{4})$.

Solution: ► *We have*

$$\frac{\partial^2}{\partial u \partial v}(e^u v \cos w) = \frac{\partial}{\partial u}(e^u \cos w) = e^u \cos w,$$

which is $\dfrac{e\sqrt{2}}{2}$ *at the desired point.* ◄

181 Example The equation $z^{xy} + (xy)^z + xy^2z^3 = 3$ defines z as an implicit function of x and y. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(1,1,1)$.

Solution: ► *We have*

$$\begin{aligned} \frac{\partial}{\partial x} z^{xy} &= \frac{\partial}{\partial x} e^{xy \log z} \\ &= \left(y \log z + \frac{xy}{z}\frac{\partial z}{\partial x} \right) z^{xy}, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} (xy)^{z} &= \frac{\partial}{\partial x} e^{z \log xy} \\ &= \left(\frac{\partial z}{\partial x} \log xy + \frac{z}{x} \right) (xy)^{z}, \end{aligned}$$

$$\frac{\partial}{\partial x} xy^2z^3 = y^2z^3 + 3xy^2z^2\frac{\partial z}{\partial x},$$

Hence, at $(1,1,1)$ *we have*

$$\frac{\partial z}{\partial x} + 1 + 1 + 3\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{1}{2}.$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial y} z^{xy} &= \frac{\partial}{\partial y} e^{xy \log z} \\ &= \left(x \log z + \frac{xy}{z}\frac{\partial z}{\partial y} \right) z^{xy}, \end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}(xy)^z &= \frac{\partial}{\partial y}e^{z\log xy}\\ &= \left(\frac{\partial z}{\partial y}\log xy + \frac{z}{y}\right)(xy)^z,\end{aligned}$$

$$\frac{\partial}{\partial y}xy^2z^3 = 2xyz^3 + 3xy^2z^2\frac{\partial z}{\partial y},$$

Hence, at $(1,1,1)$ *we have*

$$\frac{\partial z}{\partial y} + 1 + 2 + 3\frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{3}{4}.$$

◀

Just like in the one-variable case, we have the following rules of differentiation. Let $A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m$ be open sets $f, g : A \to \mathbb{R}^m, \alpha \in \mathbb{R}$, be differentiable on A, $h : B \to \mathbb{R}^l$ be differentiable on B, and $f(A) \subseteq B$. Then we have

- **Addition Rule:** $\mathscr{D}_{\mathrm{x}}((f + \alpha g)) = \mathscr{D}_{\mathrm{x}}(f) + \alpha\mathscr{D}_{\mathrm{x}}(g)$.
- **Chain Rule:** $\mathscr{D}_{\mathrm{x}}((h \circ f)) = \left(\mathscr{D}_{f(\mathrm{x})}(h)\right) \circ (\mathscr{D}_{\mathrm{x}}(f))$.

Since composition of linear mappings expressed as matrices is matrix multiplication, the Chain Rule takes the alternative form when applied to the Jacobi matrix.

$$(h \circ f)' = (h' \circ f)(f'). \tag{2.1}$$

182 Example Let

$$f(u, v) = \begin{bmatrix} ue^v \\ u + v \\ uv \end{bmatrix},$$

$$h(x, y) = \begin{bmatrix} x^2 + y \\ y + z \end{bmatrix}.$$

Find $(f \circ h)'(x, y)$.

Solution: ► *We have*

$$f'(u, v) = \begin{bmatrix} e^v & ue^v \\ 1 & 1 \\ v & u \end{bmatrix},$$

and

$$h'(x, y) = \begin{bmatrix} 2x & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Observe also that

$$f'(h(x, y)) = \begin{bmatrix} e^{y+z} & (x^2 + y)e^{y+z} \\ 1 & 1 \\ y + z & x^2 + y \end{bmatrix}.$$

Hence

$$\begin{aligned}(f\circ h)'(x,y) &= f'(h(x,y))h'(x,y)\\ &= \begin{bmatrix} e^{y+z} & (x^2+y)e^{y+z}\\ 1 & 1\\ y+z & x^2+y\end{bmatrix}\begin{bmatrix}2x & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\\ &= \begin{bmatrix} 2xe^{y+z} & (1+x^2+y)e^{y+z} & (x^2+y)e^{y+z}\\ 2x & 2 & 1\\ 2xy+2xz & x^2+2y+z & x^2+y\end{bmatrix}.\end{aligned}$$

◀

183 Example Let

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(u,v)=u^2+e^v,$$

$$u,v:\mathbb{R}^3\to\mathbb{R}\quad u(x,y)=xz,\ v(x,y)=y+z.$$

Put $h(x,y)=f\begin{bmatrix}u(x,y,z)\\ v(x,y,z)\end{bmatrix}$. Find the partial derivatives of h.

Solution: ► *Put* $g:\mathbb{R}^3\to\mathbb{R}^2, g(x,y)=\begin{bmatrix}u(x,y)\\ v(x,y\end{bmatrix}=\begin{bmatrix}xz\\ y+z\end{bmatrix}$. *Observe that* $h=f\circ g$. *Now,*

$$g'(x,y)=\begin{bmatrix}z & 0 & x\\ 0 & 1 & 1\end{bmatrix},$$

$$f'(u,v)=\begin{bmatrix}2u & e^v\end{bmatrix},$$

$$f'(h(x,y))=\begin{bmatrix}2xz & e^{y+z}\end{bmatrix}.$$

Thus

$$\begin{aligned}\begin{bmatrix}\frac{\partial h}{\partial x}(x,y) & \frac{\partial h}{\partial y}(x,y) & \frac{\partial h}{\partial z}(x,y)\end{bmatrix} &= h'(x,y)\\ &= (f'(g(x,y)))(g'(x,y))\\ &= \begin{bmatrix}2xz & e^{y+z}\end{bmatrix}\begin{bmatrix}z & 0 & x\\ 0 & 1 & 1\end{bmatrix}.\\ &= \begin{bmatrix}2xz^2 & e^{y+z} & 2x^2z+e^{y+z}\end{bmatrix}\end{aligned}$$

Equating components, we obtain

$$\frac{\partial h}{\partial x}(x,y)=2xz^2,$$

$$\frac{\partial h}{\partial y}(x,y)=e^{y+z},$$

$$\frac{\partial h}{\partial z}(x,y)=2x^2z+e^{y+z}.$$

◀

Under certain conditions we may differentiate under the integral sign.

184 Theorem (Differentiation under the integral sign) Let $f\colon [a,b]\times Y \to \mathbb{R}$ be a function, with $[a,b]$ being a closed interval, and Y being a closed and bounded subset of $\mathbb{R}$. Suppose that both $f(x,y)$ and $\frac{\partial}{\partial x}f(x,y)$ are continuous in the variables x and y jointly. Then $\int_Y f(x.y)\,dy$ exists as a continuously differentiable function of x on $[a,b]$, with derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_Y f(x,y)\,\mathrm{d}y = \int_Y \frac{\partial}{\partial x}f(x,y)\,\mathrm{d}y.$$

185 Example Prove that

$$F(x) = \int_0^{\pi/2} \log(\sin^2\theta + x^2\cos^2\theta)\mathrm{d}\theta = \pi\log\frac{x+1}{2}.$$

Solution: ► *Differentiating under the integral,*

$$\begin{aligned} F'(x) &= \int_0^{\pi/2}\frac{\partial}{\partial x}\log(\sin^2\theta + x^2\cos^2\theta)\mathrm{d}\theta \\ &= 2x\int_0^{\pi/2}\frac{\cos^2\theta}{\sin^2\theta + x^2\cos^2\theta}\mathrm{d}\theta \end{aligned}$$

. The above implies that

$$\begin{aligned} \frac{(x^2-1)}{2x}\cdot F'(x) &= \int_0^{\pi/2}\frac{(x^2-1)\cos^2\theta}{\sin^2\theta + x^2\cos^2\theta}\mathrm{d}\theta \\ &= \int_0^{\pi/2}\frac{x^2\cos^2\theta + \sin^2\theta - 1}{\sin^2\theta + x^2\cos^2\theta}\mathrm{d}\theta \\ &= \frac{\pi}{2} - \int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sin^2\theta + x^2\cos^2\theta} \\ &= \frac{\pi}{2} - \int_0^{\pi/2}\frac{\sec^2\theta\mathrm{d}\theta}{\tan^2\theta + x^2} \\ &= \frac{\pi}{2} - \frac{1}{x}\arctan\frac{\tan\theta}{x}\Big|_0^{\pi/2} \\ &= \frac{\pi}{2} - \frac{\pi}{2x}, \end{aligned}$$

which in turn implies that for $x > 0$, $x \neq 1$,

$$F'(x) = \frac{2x}{x^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2x}\right) = \frac{\pi}{x+1}.$$

For $x = 1$ one sees immediately that $F'(1) = 2\int_0^{\pi/2}\cos^2\theta\mathrm{d}\theta = \frac{\pi}{2}$, agreeing with the formula. Now,

$$F'(x) = \frac{\pi}{x+1} \implies F(x) = \pi\log(x+1) + C.$$

Since $F(1) = \int_0^{\pi/2}\log 1\mathrm{d}\theta = 0$, we gather that $C = -\pi\log 2$. Finally thus

$$F(x) = \pi\log(x+1) - \pi\log 2 = \pi\log\frac{x+1}{2}.$$

◄

Under certain conditions, the interval of integration in the above theorem need not be compact.

186 Example Given that $\int_0^{+\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$, compute $\int_0^{+\infty}\frac{\sin^2 x}{x^2}\mathrm{d}x$.

Solution: ► *Put* $I(a) = \int_0^{+\infty} \frac{\sin^2 ax}{x^2} dx$*, with* $a \geq 0$*. Differentiating both sides with respect to* a*, and making the substitution* $u = 2ax$*,*

$$\begin{aligned} I'(a) &= \int_0^{+\infty} \frac{2x \sin ax \cos ax}{x^2} dx \\ &= \int_0^{+\infty} \frac{\sin 2ax}{x} dx \\ &= \int_0^{+\infty} \frac{\sin u}{u} du \\ &= \frac{\pi}{2}. \end{aligned}$$

Integrating each side gives

$$I(a) = \frac{\pi}{2} a + C.$$

Since $I(0) = 0$*, we gather that* $C = 0$*. The desired integral is* $I(1) = \frac{\pi}{2}$*.* ◄

Homework

Problem 2.5.1 Let $f : [0; +\infty[\times]0; +\infty[\to \mathbb{R}$, $f(r, t) = t^n e^{-r^2/4t}$, where n is a constant. Determine n such that

$$\frac{\partial f}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right).$$

Problem 2.5.2 Let

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = \min(x, y^2).$$

Find $\frac{\partial f(x, y)}{\partial x}$ and $\frac{\partial f(x, y)}{\partial y}$.

Problem 2.5.3 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x, y) = \begin{bmatrix} xy^2 \\ x^2 y \end{bmatrix}, \qquad g(x, y, z) = \begin{bmatrix} x - y + 2z \\ xy \end{bmatrix}.$$

Compute $(f \circ g)'(1, 0, 1)$, if at all defined. If undefined, explain. Compute $(g \circ f)'(1, 0)$, if at all defined. If undefined, explain.

Problem 2.5.4 Let $f(x, y) = \begin{bmatrix} xy \\ x + y \end{bmatrix}$ and $g(x, y) = \begin{bmatrix} x - y \\ x^2 y^2 \\ x + y \end{bmatrix}$ Find $(g \circ f)'(0, 1)$.

Problem 2.5.5 Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$ is a constant. Find the partial derivatives with respect to x and y of

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \int_a^{x^2 y} g(t)\, dt.$$

Problem 2.5.6 Given that $\int_0^b \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{b}{a}$, evaluate $\int_0^b \frac{dx}{(x^2 + a^2)^2}$.

Problem 2.5.7 Prove that

$$\int_0^{+\infty} \frac{\arctan ax - \arctan x}{x} dx = \frac{\pi}{2} \log \pi.$$

Problem 2.5.8 Assuming that the equation $xy^2 + 3z = \cos z^2$ defines z implicitly as a function of x and y, find $\frac{\partial z}{\partial x}$.

Problem 2.5.9 If $w = e^{uv}$ and $u = r + s$, $v = rs$, determine $\frac{\partial w}{\partial r}$.

Problem 2.5.10 Let z be an implicitly-defined function of x and y through the equation $(x + z)^2 + (y + z)^2 = 8$. Find $\frac{\partial z}{\partial x}$ at $(1, 1, 1)$.

2.6 Gradients and Directional Derivatives

A function

$$f : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^m \\ \mathbf{x} & \mapsto & f(\mathbf{x}) \end{array}$$

is called a *vector field*. If $m = 1$, it is called a *scalar field*.

187 Definition Let

$$f : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R} \\ \mathbf{x} & \mapsto & f(\mathbf{x}) \end{array}$$

be a scalar field. The *gradient* of f is the vector defined and denoted by

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{bmatrix}.$$

The *gradient operator* is the operator

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}.$$

188 Theorem Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}$ be a scalar field, and assume that f is differentiable in A. Let $K \in \mathbb{R}$ be a constant. Then $\nabla f(\mathbf{x})$ is orthogonal to the surface implicitly defined by $f(\mathbf{x}) = K$.

Proof: *Let*

$$\mathbf{c} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^n \\ t & \mapsto & \mathbf{c}(t) \end{array}$$

be a curve lying on this surface. Choose t_0 so that $\mathbf{c}(t_0) = \mathbf{x}$. Then

$$(f \circ \mathbf{c})(t_0) = f(\mathbf{c}(t)) = K,$$

and using the chain rule

$$f'(\mathbf{c}(t_0))\mathbf{c}'(t_0) = 0,$$

which translates to

$$(\nabla f(\mathbf{x}))\bullet(\mathbf{c}'(t_0)) = 0.$$

Since $\mathbf{c}'(t_0)$ is tangent to the surface and its dot product with $\nabla f(\mathbf{x})$ is 0, we conclude that $\nabla f(\mathbf{x})$ is normal to the surface. ❑

☞ *Let θ be the angle between $\nabla f(\mathbf{x})$ and $\mathbf{c}'(t_0)$. Since*

$$|(\nabla f(\mathbf{x}))\bullet(\mathbf{c}'(t_0))| = ||\nabla f(\mathbf{x})||||\mathbf{c}'(t_0)|| \cos\theta,$$

$\nabla f(\mathbf{x})$ is the direction in which f is changing the fastest.

189 Example Find a unit vector normal to the surface $x^3 + y^3 + z = 4$ at the point $(1, 1, 2)$.

Solution: ► *Here $f(x, y, z) = x^3 + y^3 + z - 4$ has gradient*

$$\nabla f(x, y, z) = \begin{bmatrix} 3x^2 \\ 3y^2 \\ 1 \end{bmatrix}$$

which at $(1,1,2)$ *is* $\begin{bmatrix}3\\3\\1\end{bmatrix}$. *Normalising this vector we obtain*

$$\begin{bmatrix}\frac{3}{\sqrt{19}}\\ \frac{3}{\sqrt{19}}\\ \frac{1}{\sqrt{19}}\end{bmatrix}.$$

◀

190 Example Find the direction of the greatest rate of increase of $f(x,y,z) = xye^z$ at the point $(2,1,2)$.

Solution: ▶ *The direction is that of the gradient vector. Here*

$$\nabla f(x,y,z) = \begin{bmatrix}ye^z\\xe^z\\xye^z\end{bmatrix}$$

which at $(2,1,2)$ *becomes* $\begin{bmatrix}e^2\\2e^2\\2e^2\end{bmatrix}$. *Normalising this vector we obtain*

$$\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\2\end{bmatrix}.$$

◀

191 Example Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x,y,z) = x + y^2 - z^2.$$

Find the equation of the tangent plane to f at $(1,2,3)$.

Solution: ▶ *A vector normal to the plane is* $\nabla f(1,2,3)$. *Now*

$$\nabla f(x,y,z) = \begin{bmatrix}1\\2y\\-2z\end{bmatrix}$$

which is

$$\begin{bmatrix}1\\4\\-6\end{bmatrix}$$

at $(1,2,3)$. *The equation of the tangent plane is thus*

$$1(x-1) + 4(y-2) - 6(z-3) = 0,$$

or

$$x + 4y - 6z = -9.$$

◀

192 Definition Let

$$f: \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ \mathbf{x} & \mapsto & f(\mathbf{x}) \end{array}$$

be a vector field with

$$f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}.$$

The *divergence* of f is defined and denoted by

$$\mathrm{div} f(\mathbf{x}) = \nabla \bullet f(\mathbf{x}) = \frac{\partial f_1}{\partial x_1}(\mathbf{x}) + \frac{\partial f_2}{\partial x_2}(\mathbf{x}) + \cdots + \frac{\partial f_n}{\partial x_n}(\mathbf{x}).$$

193 Example If $f(x, y, z) = (x^2, y^2, ye^{z^2})$ then

$$\mathrm{div} f(\mathbf{x}) = 2x + 2y + 2yze^{z^2}.$$

194 Definition Let $g_k : \mathbb{R}^n \to \mathbb{R}^n$, $1 \leq k \leq n-2$ be vector fields with $g_i = (g_{i1}, g_{i2}, \ldots, g_{in})$. Then the *curl* of $(g_1, g_2, \ldots, g_{n-2})$

$$\mathrm{curl}(g_1, g_2, \ldots, g_{n-2})(\mathbf{x}) = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \cdots & \dfrac{\partial}{\partial x_n} \\ g_{11}(\mathbf{x}) & g_{12}(\mathbf{x}) & \cdots & g_{1n}(\mathbf{x}) \\ g_{21}(\mathbf{x}) & g_{22}(\mathbf{x}) & \cdots & g_{2n}(\mathbf{x}) \\ \vdots & \vdots & \vdots & \vdots \\ g_{(n-2)1}(\mathbf{x}) & g_{(n-2)2}(\mathbf{x}) & \cdots & g_{(n-2)n}(\mathbf{x}) \end{bmatrix}.$$

195 Example If $f(x, y, z) = (x^2, y^2, ye^{z^2})$ then

$$\mathrm{curl} f((\mathbf{x}, \mathbf{y}, \mathbf{z})) = \nabla \times f(x, y, z) = (e^{z^2})\mathbf{i}.$$

196 Example If $f(x, y, z, w) = (e^{xyz}, 0, 0, w^2)$, $g(x, y, z, w) = (0, 0, z, 0)$ then

$$\mathrm{curl}(f, g)(x, y, z, w) = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_4} \\ e^{xyz} & 0 & 0 & w^2 \\ 0 & 0 & z & 0 \end{bmatrix} = (xz^2e^{xyz})\mathbf{e}_4.$$

197 Definition Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}$ be a scalar field, and assume that f is differentiable in A. Let $\overrightarrow{\mathbf{v}} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ be such that $\mathbf{x} + t\overrightarrow{\mathbf{v}} \in A$ for sufficiently small $t \in \mathbb{R}$. Then the *directional derivative of f in the direction of $\overrightarrow{\mathbf{v}}$ at the point* $\mathbf{x}$ is defined and denoted by

$$\mathscr{D}_{\overrightarrow{\mathbf{v}}} f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\overrightarrow{\mathbf{v}}) - f(\mathbf{x})}{t}.$$

☞ *Some authors require that the vector $\overrightarrow{\mathbf{v}}$ in definition 197 be a unit vector.*

198 Theorem Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}$ be a scalar field, and assume that f is differentiable in A. Let $\overrightarrow{v} \in \mathbb{R}^n \setminus \{\overrightarrow{0}\}$ be such that $\overrightarrow{x} + t\overrightarrow{v} \in A$ for sufficiently small $t \in \mathbb{R}$. Then the *directional derivative of f in the direction of $\overrightarrow{v}$ at the point $\overrightarrow{x}$* is given by

$$\nabla f(\mathrm{x}) \bullet \overrightarrow{v}.$$

199 Example Find the directional derivative of $f(x, y, z) = x^3 + y^3 - z^2$ in the direction of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution: ► *We have*

$$\nabla f(x, y, z) = \begin{bmatrix} 3x^2 \\ 3y^2 \\ -2z \end{bmatrix}$$

and so

$$\nabla f(x, y, z) \bullet \overrightarrow{v} = 3x^2 + 6y^2 - 6z.$$

◄

Homework

Problem 2.6.1 Let $f(x, y, z) = xe^{yz}$. Find

$$(\nabla f)(2, 1, 1).$$

Problem 2.6.2 Let $f(x, y, z) = \begin{bmatrix} xz \\ e^{xy} \\ z \end{bmatrix}$. Find

$$(\nabla \times f)(2, 1, 1).$$

Problem 2.6.3 Find the tangent plane to the surface $\frac{x^2}{2} - y^2 - z^2 = 0$ at the point $(2, -1, 1)$.

Problem 2.6.4 Find the point on the surface

$$x^2 + y^2 - 5xy + xz - yz = -3$$

for which the tangent plane is $x - 7y = -6$.

Problem 2.6.5 Find a vector pointing in the direction in which $f(x, y, z) = 3xy - 9xz^2 + y$ increases most rapidly at the point $(1, 1, 0)$.

Problem 2.6.6 Let $\mathscr{D}_{\overrightarrow{u}} f(x, y)$ denote the directional derivative of f at (x, y) in the direction of the unit vector $\overrightarrow{u}$. If $\nabla f(1, 2) = 2\overrightarrow{i} - \overrightarrow{j}$, find $\mathscr{D}_{(\frac{3}{5}, \frac{4}{5})} f(1, 2)$.

Problem 2.6.7 Use a linear approximation of the function $f(x, y) = e^{x \cos 2y}$ at (0, 0) to estimate $f(0.1, 0.2)$.

Problem 2.6.8 Prove that

$$\nabla \bullet (\mathrm{u} \times \mathrm{v}) = \mathrm{v} \bullet (\nabla \times \mathrm{u}) - \mathrm{u} \bullet (\nabla \times \mathrm{v}).$$

Problem 2.6.9 Find the point on the surface

$$2x^2 + xy + y^2 + 4x + 8y - z + 14 = 0$$

for which the tangent plane is $4x + y - z = 0$.

Problem 2.6.10 Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field, and let $\mathrm{U}, \mathrm{V} : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields. Prove that

1. $\nabla \bullet \phi \mathrm{V} = \phi \nabla \bullet \mathrm{V} + \mathrm{V} \bullet \nabla \phi$
2. $\nabla \times \phi \mathrm{V} = \phi \nabla \times \mathrm{V} + (\nabla \phi) \times \mathrm{V}$
3. $\nabla \times (\nabla \phi) = \overrightarrow{0}$
4. $\nabla \bullet (\nabla \times \mathrm{V}) = 0$
5. $\nabla(\mathrm{U} \bullet \mathrm{V}) = (\mathrm{U} \bullet \nabla)\mathrm{V} + (\mathrm{V} \bullet \nabla)\mathrm{U} + \mathrm{U} \times (\nabla \times \mathrm{V}) + {}+\mathrm{V} \times (\nabla \times \mathrm{U})$

Problem 2.6.11 Find the angles made by the gradient of $f(x, y) = x^{\sqrt{3}} + y$ at the point $(1, 1)$ with the coordinate axes.

2.7 Levi-Civitta and Einstein

☞ *In this section, unless otherwise noted, we are dealing in the space $\mathbb{R}^3$ and so, subscripts are in the set $\{1, 2, 3\}$.*

200 Definition (Einstein's Summation Convention) In any expression containing subscripted variables appearing twice (and only twice) in any term, the subscripted variables are assumed to be summed over.

☞ *In order to emphasise that we are using Einstein's convention, we will enclose any terms under consideration with* $\olessthan \cdot \ogreaterthan$.

201 Example Using Einstein's Summation convention, the dot product of two vectors $\overrightarrow{x} \in \mathbb{R}^n$ and $\overrightarrow{y} \in \mathbb{R}^n$ can be written as

$$\overrightarrow{x} \bullet \overrightarrow{y} = \sum_{i=1}^{n} x_i y_i = \olessthan x_t y_t \ogreaterthan .$$

202 Example Given that a_i, b_j, c_k, d_l are the components of vectors in $\mathbb{R}^3$, $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}$ respectively, what is the meaning of

$$\olessthan a_i b_i c_k d_k \ogreaterthan ?$$

Solution: ► *We have*

$$\olessthan a_i b_i c_k d_k \ogreaterthan = \sum_{i=1}^{3} a_i b_i \olessthan c_k d_k \ogreaterthan = \overrightarrow{a} \bullet \overrightarrow{b} \olessthan c_k d_k \ogreaterthan = \overrightarrow{a} \bullet \overrightarrow{b} \sum_{k=1}^{3} c_k d_k = (\overrightarrow{a} \bullet \overrightarrow{b})(\overrightarrow{c} \bullet \overrightarrow{d}).$$

◄

203 Example Using Einstein's Summation convention, the ij-th entry $(AB)_{ij}$ of the product of two matrices $A \in \mathrm{M}_{m \times n}(\mathbb{R})$ and $B \in \mathrm{M}_{n \times r}(\mathbb{R})$ can be written as

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = \olessthan A_{it} B_{tj} \ogreaterthan .$$

204 Example Using Einstein's Summation convention, the trace $\mathrm{tr}\,(A)$ of a square matrix $A \in \mathrm{M}_{n \times n}(\mathbb{R})$ is $\mathrm{tr}\,(A) = \sum_{t=1}^{n} A_{tt} = \olessthan A_{tt} \ogreaterthan$.

205 Example Demonstrate, via Einstein's Summation convention, that if A, B are two $n \times n$ matrices, then

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA) .$$

Solution: ► *We have*

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,((AB)_{ij}) = \mathrm{tr}\,(\olessthan A_{ik} B_{kj} \ogreaterthan) = \olessthan \olessthan A_{tk} B_{kt} \ogreaterthan \ogreaterthan ,$$

and

$$\mathrm{tr}\,(BA) = \mathrm{tr}\,((BA)_{ij}) = \mathrm{tr}\,(\olessthan B_{ik} A_{kj} \ogreaterthan) = \olessthan \olessthan B_{tk} A_{kt} \ogreaterthan \ogreaterthan ,$$

from where the assertion follows, since the indices are dummy variables and can be exchanged. ◄

206 Definition (Kroenecker's Delta) The symbol δ_{ij} is defined as follows:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

207 Example It is easy to see that $\olessthan \delta_{ik} \delta_{kj} \ogreaterthan = \sum_{k=1}^{3} \delta_{ik} \delta_{kj} = \delta_{ij}$.

208 Example We see that

$$\circleddash\delta_{ij}a_ib_j\circleddash = \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}a_ib_j = \sum_{k=1} a_kb_k = \overrightarrow{a}\bullet\overrightarrow{b}.$$

Recall that a *permutation* of distinct objects is a reordering of them. The $3! = 6$ permutations of the index set $\{1, 2, 3\}$ can be classified into *even* or *odd*. We start with the identity permutation 123 and say it is even. Now, for any other permutation, we will say that it is even if it takes an even number of transpositions (switching only two elements in one move) to regain the identity permutation, and odd if it takes an odd number of transpositions to regain the identity permutation. Since

$$231 \to 132 \to 123, \quad 312 \to 132 \to 123,$$

the permutations 123 (identity), 231, and 312 are even. Since

$$132 \to 123, \quad 321 \to 123, \quad 213 \to 123,$$

the permutations 132, 321, and 213 are odd.

209 Definition (Levi-Civitta's Alternating Tensor) The symbol ε_{jkl} is defined as follows:

$$\varepsilon_{jkl} = \begin{cases} 0 & \text{if } \{j,k,l\} \neq \{1,2,3\} \\ -1 & \text{if } \begin{pmatrix} 1 & 2 & 3 \\ j & k & l \end{pmatrix} \text{ is an odd permutation} \\ +1 & \text{if } \begin{pmatrix} 1 & 2 & 3 \\ j & k & l \end{pmatrix} \text{ is an even permutation} \end{cases}$$

In particular, if one subindex is repeated we have $\varepsilon_{rrs} = \varepsilon_{rsr} = \varepsilon_{srr} = 0$. Also,

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \qquad \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1.$$

210 Example Using the Levi-Civitta alternating tensor and Einstein's summation convention, the cross product can also be expressed, if $\overrightarrow{i} = \overrightarrow{e_1}$, $\overrightarrow{j} = \overrightarrow{e_2}$, $\overrightarrow{k} = \overrightarrow{e_3}$, then

$$\overrightarrow{x} \times \overrightarrow{y} = \circleddash\varepsilon_{jkl}(a_kb_l)\overrightarrow{e_j}\circleddash.$$

211 Example If $A = [a_{ij}]$ is a 3×3 matrix, then, using the Levi-Civitta alternating tensor,

$$\det A = \circleddash\varepsilon_{ijk}a_{1i}a_{2j}a_{3k}\circleddash.$$

212 Example Let $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$ be vectors in $\mathbb{R}^3$. Then

$$\overrightarrow{x} \bullet (\overrightarrow{y} \times \overrightarrow{z}) = \circleddash x_i(\overrightarrow{y} \times \overrightarrow{z})_i\circleddash = \circleddash x_i\varepsilon_{ikl}(y_kz_l)\circleddash.$$

Homework

Problem 2.7.1 Let $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$ be vectors in $\mathbb{R}^3$. Demonstrate that

$$\circleddash x_iy_iz_j\circleddash = (\overrightarrow{x}\bullet\overrightarrow{y})\overrightarrow{z}.$$

2.8 Extrema

We now turn to the problem of finding maxima and minima for vector functions. As in the one-variable case, the derivative will provide us with information about the extrema, and the "second derivative" will provide us with information about the nature of these extreme points.

To define an analogue for the second derivative, let us consider the following. Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be differentiable on A. We know that for fixed $\mathrm{x}_0 \in A$, $\mathscr{D}_{\mathrm{x}_0}(f)$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. This means that we have a function

$$T : \begin{array}{ccc} A & \to & \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m) \\ \mathrm{x} & \mapsto & \mathscr{D}_{\mathrm{x}}(f) \end{array},$$

where $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Hence, if we differentiate T at x_0 again, we obtain a linear transformation $\mathscr{D}_{\mathrm{x}_0}(T) = \mathscr{D}_{\mathrm{x}_0}(\mathscr{D}_{\mathrm{x}_0}(f)) = \mathscr{D}^2_{\mathrm{x}_0}(f)$ from $\mathbb{R}^n$ to $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$. Hence, given $\mathrm{x}_1 \in \mathbb{R}^n$, we have $\mathscr{D}^2_{\mathrm{x}_0}(f)(\mathrm{x}_1) \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$. Again, this means that given $\mathrm{x}_2 \in \mathbb{R}^n$, $\mathscr{D}^2_{\mathrm{x}_0}(f)(\mathrm{x}_1))(\mathrm{x}_2) \in \mathbb{R}^m$. Thus the function

$$B_{\mathrm{x}_0} : \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m) \\ (\mathrm{x}_1, \mathrm{x}_2) & \mapsto & \mathscr{D}^2_{\mathrm{x}_0}(f)(\mathrm{x}_1, \mathrm{x}_2) \end{array}$$

is well defined, and linear in each variable x_1 and x_2, that is, it is a *bilinear* function.

Just as the Jacobi matrix was a handy tool for finding a matrix representation of $\mathscr{D}_{\mathrm{x}}(f)$ in the natural bases, when f maps into $\mathbb{R}$, we have the following analogue representation of the second derivative.

213 Theorem Let $A \subseteq \mathbb{R}^n$ be an open set, and $f : A \to \mathbb{R}$ be twice differentiable on A. Then the matrix of $\mathscr{D}^2_{\mathrm{x}}(f) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with respect to the standard basis is given by the *Hessian matrix:*

$$\mathscr{H}_x f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1}(\mathrm{x}) & \dfrac{\partial^2 f}{\partial x_1 \partial x_2}(\mathrm{x}) & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(\mathrm{x}) \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1}(\mathrm{x}) & \dfrac{\partial^2 f}{\partial x_2 \partial x_2}(\mathrm{x}) & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n}(\mathrm{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1}(\mathrm{x}) & \dfrac{\partial^2 f}{\partial x_n \partial x_2}(\mathrm{x}) & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_n}(\mathrm{x}) \end{bmatrix}$$

214 Example Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = xy^2 z^3.$$

Then

$$\mathscr{H}_{(x,y,z)} f = \begin{bmatrix} 0 & 2yz^3 & 3y^2z^2 \\ 2yz^3 & 2xz^3 & 6xyz^2 \\ 3y^2z^2 & 6xyz^2 & 6xy^2z \end{bmatrix}$$

From the preceding example, we notice that the Hessian is symmetric, as the mixed partial derivatives $\dfrac{\partial^2}{\partial x \partial y} f = \dfrac{\partial^2}{\partial y \partial x} f$, etc., are equal. This is no coincidence, as guaranteed by the following theorem.

215 Theorem Let $A \subseteq \mathbb{R}^n$ be an open set, and $f : A \to \mathbb{R}$ be twice differentiable on A. If $\mathscr{D}^2_{\mathrm{x}_0}(f)$ is continuous, then $\mathscr{D}^2_{\mathrm{x}_0}(f)$ is symmetric, that is, $\forall(\mathrm{x}_1, \mathrm{x}_2) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$\mathscr{D}^2_{\mathrm{x}_0}(f)(\mathrm{x}_1, \mathrm{x}_2) = \mathscr{D}^2_{\mathrm{x}_0}(f)(\mathrm{x}_2, \mathrm{x}_1).$$

We are now ready to study extrema in several variables. The basic theorems resemble those of one-variable calculus. First, we make some analogous definitions.

216 Definition Let $A \subseteq \mathbb{R}^n$ be an open set, and $f : A \to \mathbb{R}$. If there is some open ball $B_{\mathbf{x}_0}(\mathbf{r})$ on which $\forall \mathbf{x} \in B_{x_0}(\mathbf{r}),\ \ f(\mathbf{x}_0) \geq f(\mathbf{x})$, we say that $f(\mathbf{x}_0)$ is a *local maximum* of f. Similarly, if there is some open ball $B_{\mathbf{x}_1}(\mathbf{r})$ on which $\forall \mathbf{x} \in B_{\mathbf{x}_0}(\mathbf{r}'),\ \ f(\mathbf{x}_1) \leq f(\mathbf{x})$, we say that $f(\mathbf{x}_1)$ is a *local maximum* of f. A point is called an *extreme point* if it is either a local minimum or local maximum. A point $\mathbf{t}$ is called a *critical point* if f is differentiable at $\mathbf{t}$ and $\mathscr{D}_{\mathbf{t}}(f) = \mathbf{0}$. A critical point which is neither a maxima nor a minima is called a *saddle point*.

217 Theorem Let $A \subseteq \mathbb{R}^n$ be an open set, and $f : A \to \mathbb{R}$ be differentiable on A. If $\mathbf{x}_0$ is an extreme point, then $\mathscr{D}_{\mathbf{x}_0}(f) = \mathbf{0}$, that is, $\mathbf{x}_0$ is a critical point. Moreover, if f is twice-differentiable with continuous second derivative and $\mathbf{x}_0$ is a critical point such that $\mathscr{H}_{\mathbf{x}_0} f$ is negative definite, then f has a local maximum at $\mathbf{x}_0$. If $\mathscr{H}_{\mathbf{x}_0} f$ is positive definite, then f has a local minimum at $\mathbf{x}_0$. If $\mathscr{H}_{\mathbf{x}_0} f$ is indefinite, then f has a saddle point. If $\mathscr{H}_{\mathbf{x}_0} f$ is semi-definite (positive or negative), the test is inconclusive.

218 Example Find the critical points of

$$f : \begin{array}{lll} \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & x^2 + xy + y^2 + 2x + 3y \end{array}$$

and investigate their nature.

Solution: ▶ *We have*

$$(\nabla f)(x, y) = \begin{bmatrix} 2x + y + 2 \\ x + 2y + 3 \end{bmatrix},$$

and so to find the critical points we solve

$$2x + y + 2 = 0,$$

$$x + 2y + 3 = 0,$$

which yields $x = -\dfrac{1}{3}, y = -\dfrac{4}{3}$. *Now,*

$$\mathscr{H}_{(x,y)} f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

which is positive definite, since $\Delta_1 = 2 > 0$ *and* $\Delta_2 = \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3 > 0$. *Thus* $\mathbf{x}_0 = \left(-\dfrac{1}{3}, -\dfrac{4}{3}\right)$ *is a relative minimum and we have*

$$-\frac{7}{3} = f\left(-\frac{1}{3}, -\frac{4}{3}\right) \leq f(x, y) = x^2 + xy + y^2 + 2x + 3y.$$

◀

219 Example Find the extrema of

$$f : \begin{array}{lll} \mathbb{R}^3 & \to & \mathbb{R} \\ (x, y, z) & \mapsto & x^2 + y^2 + 3z^2 - xy + 2xz + yz \end{array}.$$

Solution: ► *We have*

$$(\nabla f)(x,y,z) = \begin{bmatrix} 2x-y+2z \\ 2y-x+z \\ 6z+2x+y \end{bmatrix},$$

which vanishes when $x=y=z=0$. *Now,*

$$\mathscr{H}_r f = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{bmatrix},$$

which is positive definite, since $\Delta_1 = 2 > 0$, $\Delta_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 > 0$, *and* $\Delta_3 = \det \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{bmatrix} = 4 > 0$. *Thus* f *has a relative minimum at* $(0,0,0)$ *and*

$$0 = f(0,0,0) \leq f(x,y,z) = x^2+y^2+3z^2-xy+2xz+yz.$$

◄

220 Example Let $f(x,y) = x^3 - y^3 + axy$, with $a \in \mathbb{R}$ a parameter. Determine the nature of the critical points of f.

Solution: ► *We have*

$$(\nabla f)(x,y) = \begin{bmatrix} 3x^2+ay \\ -3y^2+ax \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 3x^2 = -ay, \quad 3y^2 = ax.$$

If $a = 0$, *then* $x = y = 0$ *and so* $(0,0)$ *is a critical point. If* $a \neq 0$ *then*

$$\begin{aligned} 3\left(3\frac{y^2}{a}\right)^2 = -ay &\implies 27y^4 = -a^3y \\ &\implies y(27y^3+a^3) = 0 \\ &\implies y(3y+a)(9y^2-3ay+a^2) = 0 \\ &\implies y = 0 \quad \text{or} \quad y = -\frac{a}{3}. \end{aligned}$$

If $y = 0$ *then* $x = 0$, *so again* $(0,0)$ *is a critical point. If* $y = -\frac{a}{3}$ *then* $x = \frac{3}{a}\cdot\left(-\frac{a}{3}\right)^2 = \frac{a}{3}$ *so* $\left(\frac{a}{3}, -\frac{a}{3}\right)$ *is a critical point.*
Now,

$$\mathcal{H}_{f(x,y)} = \begin{bmatrix} 6x & a \\ a & -6y \end{bmatrix} \implies \Delta_1 = 6x, \quad \Delta_2 = -36xy - a^2.$$

At $(0,0)$, $\Delta_1 = 0$, $\Delta_2 = -a^2$. *If* $a \neq 0$ *then there is a saddle point. At* $\left(\frac{a}{3}, -\frac{a}{3}\right)$, $\Delta_1 = 2a$, $\Delta_2 = 3a^2$, *whence* $\left(\frac{a}{3}, -\frac{a}{3}\right)$ *will be a local minimum if* $a > 0$ *and a local maximum if* $a < 0$. ◄

Homework

Problem 2.8.1 Determine the nature of the critical points of $f(x, y) = y^2 - 2x^2y + 4x^3 + 20x^2$.

Problem 2.8.2 Determine the nature of the critical points of $f(x, y) = (x - 2)^2 + 2y^2$.

Problem 2.8.3 Determine the nature of the critical points of $f(x, y) = (x - 2)^2 - 2y^2$.

Problem 2.8.4 Determine the nature of the critical points of $f(x, y) = x^4 + 4xy - 2y^2$.

Problem 2.8.5 Determine the nature of the critical points of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Problem 2.8.6 Determine the nature of the critical points of $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

Problem 2.8.7 Determine the nature of the critical points of $f(x, y) = x^4 + y^4 - 2(x - y)^2$.

Problem 2.8.8 Determine the nature of the critical points of

$$f(x, y, z) = 4x^2z - 2xy - 4x^2 - z^2 + y.$$

Problem 2.8.9 Find the extrema of

$$f(x, y, z) = x^2 + y^2 + z^2 + xyz.$$

Problem 2.8.10 Find the extrema of $f(x, y, z) = x^2y + y^2z + 2x - z$.

Problem 2.8.11 Determine the nature of the critical points of

$$f(x, y, z) = 4xyz - x^4 - y^4 - z^4.$$

Problem 2.8.12 Determine the nature of the critical points of $f(x, y, z) = xyz(4 - x - y - z)$.

Problem 2.8.13 Determine the nature of the critical points of

$$g(x, y, z) = xyze^{-x^2-y^2-z^2}.$$

Problem 2.8.14 Let $f(x, y) = \int_{y^2-x}^{x^2+y} g(t)\mathrm{d}t$, where g is a continuously differentiable function defined over all real numbers and $g(0) = 0, g'(0) \neq 0$. Prove that $(0, 0)$ is a saddle point for f.

Problem 2.8.15 Find the minimum of

$$F(x, y) = (x - y)^2 + \left(\frac{\sqrt{144 - 16x^2}}{3} - \sqrt{4 - y^2}\right)^2,$$

for $-3 \leq x \leq 3$, $-2 \leq y \leq 2$.

2.9 Lagrange Multipliers

In some situations we wish to optimise a function given a set of constraints. For such cases, we have the following.

221 Theorem Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be functions whose respective derivatives are continuous. Let $g(\mathrm{x}_0) = c_0$ and let $S = g^{-1}(c_0)$ be the level set for g with value c_0, and assume $\nabla g(\mathrm{x}_0) \neq 0$. If the restriction of f to S has an extreme point at x_0, then $\exists\lambda \in \mathbb{R}$ such that

$$\nabla f(\mathrm{x}_0) = \lambda\nabla g(\mathrm{x}_0).$$

☞ *The above theorem only locates extrema, it does not say anything concerning the nature of the critical points found.*

222 Example Optimise $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$ given that $x^2 + y^2 = 1$.

Solution: ► *Let $g(x, y) = x^2 + y^2 - 1$. We solve*

$$\nabla f \begin{bmatrix} x \\ y \end{bmatrix} = \lambda\nabla g \begin{bmatrix} x \\ y \end{bmatrix}$$

for x, y, λ. This requires

$$\begin{bmatrix} 2x \\ -2y \end{bmatrix} = \begin{bmatrix} 2x\lambda \\ 2y\lambda \end{bmatrix}.$$

From $2x = 2x\lambda$ *we get either* $x = 0$ *or* $\lambda = 1$. *If* $x = 0$ *then* $y = \pm 1$ *and* $\lambda = -1$. *If* $\lambda = 1$, *then* $y = 0, x = \pm 1$. *Thus the potential critical points are* $(\pm 1, 0)$ *and* $(0, \pm 1)$. *If* $x^2 + y^2 = 1$ *then*

$$f(x, y) = x^2 - (1 - x^2) = 2x^2 - 1 \geq -1,$$

and

$$f(x, y) = 1 - y^2 - y^2 = 1 - 2y^2 \leq 1.$$

Thus $(\pm 1, 0)$ *are maximum points and* $(0, \pm 1)$ *are minimum points.* ◀

223 Example Find the maximum and the minimum points of $f(x, y) = 4x+3y$, subject to the constraint $x^2 + 4y^2 = 4$, using Lagrange multipliers.

Solution: ▶ *Putting* $g(x, y) = x^2 + 4y^2 - 4$ *we have*

$$\nabla f(x, y) = \lambda \nabla g(x, y) \implies \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 8y \end{bmatrix}.$$

Thus $4 = 2\lambda x, \ \ 3 = 8\lambda y$. *Clearly then* $\lambda \neq 0$. *Upon division we find* $\dfrac{x}{y} = \dfrac{16}{3}$. *Hence*

$$x^2 + 4y^2 = 4 \implies \frac{256}{9}y^2 + 4y^2 = 4 \implies y = \pm\frac{3}{\sqrt{73}}, x = \pm\frac{16}{\sqrt{73}.}$$

The maximum is clearly then

$$4\left(\frac{16}{\sqrt{73}}\right) + 3\left(\frac{3}{\sqrt{73}}\right) = \sqrt{73},$$

and the minimum is $-\sqrt{73}$. ◀

224 Example Let $a > 0, b > 0, c > 0$. Determine the maximum and minimum values of $f(x, y, z) = \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c}$ on the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Solution: ▶ *We use Lagrange multipliers. Put* $g(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$. *Then*

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \iff \begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix} = \lambda \begin{bmatrix} 2x/a^2 \\ 2y/b^2 \\ 2z/c^2 \end{bmatrix}.$$

It follows that $\lambda \neq 0$. *Hence* $x = \dfrac{a}{2\lambda}, y = \dfrac{b}{2\lambda}, z = \dfrac{c}{2\lambda}$. *Since* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, *we deduce* $\dfrac{3}{4\lambda^2} = 1$ *or* $\lambda = \pm\dfrac{\sqrt{3}}{2}$. *Since* a, b, c *are positive,* f *will have a maximum when all* x, y, z *are positive and a minimum when all* x, y, z *are negative. Thus the maximum is when*

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}},$$

and

$$f(x, y, z) \leq \frac{3}{\sqrt{3}} = \sqrt{3}$$

and the minimum is when

$$x = -\frac{a}{\sqrt{3}}, y = -\frac{b}{\sqrt{3}}, z = -\frac{c}{\sqrt{3}},$$

and

$$f(x,y,z) \geq -\frac{3}{\sqrt{3}} = -\sqrt{3}.$$

Aliter: *Using the CBS Inequality,*

$$\left|\frac{x}{a}\cdot 1+\frac{y}{b}\cdot 1+\frac{z}{c}\cdot 1\right| \leq \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{1/2}\left(1^2+1^2+1^2\right)^{1/2} = (1)\sqrt{3} \implies -\sqrt{3} \leq \frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leq \sqrt{3}.$$

◀

225 Example Let $a > 0, b > 0, c > 0$. Determine the maximum volume of the parallelepiped with sides parallel to the axes that can be enclosed inside the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Solution: ▶ *Let $2x$, $2y$, $2z$, be the dimensions of the box. We must maximise $f(x,y,z) = 8xyz$ subject to the constraint $g(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$. Using Lagrange multipliers,*

$$\nabla f(x,y,z) = \lambda\nabla g(x,y,z) \iff \begin{bmatrix} 8yz \\ 8xz \\ 8xy \end{bmatrix} = \lambda \begin{bmatrix} 2x/a^2 \\ 2y/b^2 \\ 2z/c^2 \end{bmatrix} \implies 4yz = \lambda\frac{x}{a^2},\ 4xz = \lambda\frac{y}{b^2},\ 4xy = \lambda\frac{z}{c^2}.$$

Multiplying the first inequality by x, the second by y, the third by z, and adding,

$$4xyz = \lambda\frac{x^2}{a^2},\ 4xyz = \lambda\frac{y^2}{b^2},\ 4xyz = \lambda\frac{z^2}{c^2}, \implies 12xyz = \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = \lambda.$$

Hence

$$\frac{\lambda}{3} = \lambda\frac{x^2}{a^2} = \lambda\frac{y^2}{b^2} = \lambda\frac{z^2}{c^2}.$$

If $\lambda = 0$, then $8xyz = 0$, which minimises the volume. If $\lambda \neq 0$, then

$$x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}, \qquad z = \frac{c}{\sqrt{3}},$$

and the maximum value is

$$8xyz \leq 8\frac{abc}{3\sqrt{3}}.$$

Aliter: *Using the AM-GM Inequality,*

$$(x^2y^2z^2)^{1/3} = (abc)^{2/3}\left(\frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{a^2}\right)^{1/3} \leq (abc)^{2/3}.\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{3} = \frac{1}{3} \implies 8xyz \leq \frac{8}{3\sqrt{3}}(abc).$$

◀

Homework

Problem 2.9.1 A closed box (with six outer faces), has fixed surface area of S square units. Find its maximum volume using Lagrange multipliers. That is, subject to the constraint $2ab+2bc+2ca = S$, you must maximise abc.

Problem 2.9.2 Consider the problem of finding the closest point P' on the plane $\Pi : ax+by+cz = d$,

a, b, c non-zero constants with $a + b + c \neq d$ to the point $P(1, 1, 1)$. In this problem, you will do this in three essentially different ways.

1. Do this by a geometric argument, arguing the the point P' closest to P on Π is on the perpendicular passing through P and P'.
2. Do this by means of Lagrange multipliers, by minimising a suitable function $f(x, y, z)$ subject to the constraint $g(x, y, z) = ax + by + cz - d$.
3. Do this considering the unconstrained extrema of a suitable function $h\left(x, y, \dfrac{d - ax - by}{c}\right)$.

Problem 2.9.3 Given that x, y are positive real numbers such that $x^4 + 81y^4 = 36$ find the maximum of $x + 3y$.

Problem 2.9.4 If x, y, z are positive real numbers such that $x^2y^3z = \dfrac{1}{6^2}$, what is the minimum value of $f(x, y, z) = 2x + 3y + z$?

Problem 2.9.5 Find the maximum and the minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $5x^2 + 6xy + 5y^2 = 8$.

Problem 2.9.6 Let $a > 0, b > 0, p > 1$. Maximise $f(x, y) = ax + by$ subject to the constraint $x^p + y^p = 1$.

Problem 2.9.7 Find the extrema of

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 4.$$

Problem 2.9.8 Find the axes of the ellipse

$$5x^2 + 8xy + 5y^2 = 9.$$

Problem 2.9.9 Optimise $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 = 2$, and $x + z = 1$.

Problem 2.9.10 Let x, y be strictly positive real numbers with $x + y = 1$. What is the maximum value of $x + \sqrt{xy}$?

Problem 2.9.11 Let a, b be positive real constants. Maximise $f(x, y) = x^a e^{-x} y^b e^{-y}$ on the triangle in $\mathbb{R}^2$ bounded by the lines $x \geq 0$, $y \geq 0$, $x + y \leq 1$.

Problem 2.9.12 Determine the extrema of $f(x, y) = \cos^2 x + \cos^2 y$ subject to the constraint $x - y = \dfrac{\pi}{4}$.

Problem 2.9.13 Determine the extrema of $f(x, y, z) = x - 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Problem 2.9.14 Find the points on the curve determined by the equations

$$x^2 + xy + y^2 - z^2 = 1, \quad x^2 + y^2 = 1$$

which are closest to the origin.

Problem 2.9.15 Does there exist a polynomial in two variables with real coefficients $p(x, y)$ such that $p(x, y) > 0$ for all x and y and that for all real numbers $c > 0$ there exists $(x_0, y_0) \in \mathbb{R}^2$ such that $p(x_0, y_0) = c$?

Problem 2.9.16 Maximise

$$f(x, y, z) = \log x + \log y + 3\log z$$

on the portion of sphere $x^2 + y^2 + z^2 = 5r^2$ which lies on the first octant. Demonstrate using this that for any positive real numbers a, b and c, there follows the inequality

$$abc^3 \leq 27\left(\frac{a + b + c}{5}\right)^5.$$

3 Integration

3.1 Differential Forms

We will now consider integration in several variables. In order to smooth our discussion, we need to consider the concept of differential forms.

226 Definition Consider n variables

$$x_1, x_2, \ldots, x_n$$

in n-dimensional space (used as the names of the axes), and let

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \in \mathbb{R}^n, \quad 1 \leq j \leq k,$$

be $k \leq n$ vectors in $\mathbb{R}^n$. Moreover, let $\{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ be a collection of k sub-indices. An *elementary k-differential form ($k > 1$) acting on the vectors* $\mathbf{a}_j$, $1 \leq j \leq k$ is defined and denoted by

$$\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k) = \det \begin{bmatrix} a_{j_1 1} & a_{j_1 2} & \cdots & a_{j_1 k} \\ a_{j_2 1} & a_{j_2 2} & \cdots & a_{j_2 k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j_k 1} & a_{j_k 2} & \cdots & a_{j_k k} \end{bmatrix}.$$

In other words, $\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$ is the $x_{j_1} x_{j_2} \ldots x_{j_k}$ component of the signed k-volume of a k-parallelotope in $\mathbb{R}^n$ spanned by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

☞ *By virtue of being a determinant, the wedge product $\wedge$ of differential forms has the following properties*

❶ **anti-commutativity:** $\mathrm{d}a \wedge \mathrm{d}b = -\mathrm{d}b \wedge \mathrm{d}a$.

❷ **linearity:** $\mathrm{d}(a + b) = \mathrm{d}a + \mathrm{d}b$.

❸ **scalar homogeneity:** *if* $\lambda \in \mathbb{R}$, *then* $\mathrm{d}\lambda a = \lambda \mathrm{d}a$.

❹ **associativity:** $(\mathrm{d}a \wedge \mathrm{d}b) \wedge \mathrm{d}c = \mathrm{d}a \wedge (\mathrm{d}b \wedge \mathrm{d}c)$.[1]

☞ *Anti-commutativity yields*

$$\mathrm{d}a \wedge \mathrm{d}a = 0.$$

227 Example Consider

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}^3.$$

Then

$$\mathrm{d}x(\mathbf{a}) = \det(1) = 1,$$

$$\mathrm{d}y(\mathbf{a}) = \det(0) = 0,$$

[1] Notice that associativity does not hold for the wedge product of *vectors*.

$$dz(\mathbf{a}) = \det(-1) = -1,$$

are the (signed) 1-volumes (that is, the length) of the projections of $\mathbf{a}$ onto the coordinate axes.

228 Example In $\mathbb{R}^3$ we have $dx \wedge dy \wedge dx = 0$, since we have a repeated variable.

229 Example In $\mathbb{R}^3$ we have

$$dx \wedge dz + 5dz \wedge dx + 4dx \wedge dy - dy \wedge dx + 12dx \wedge dx = -4dx \wedge dz + 5dx \wedge dy.$$

☞ *In order to avoid redundancy we will make the convention that if a sum of two or more terms have the same differential form up to permutation of the variables, we will simplify the summands and express the other differential forms in terms of the one differential form whose indices appear in increasing order.*

230 Definition A 0-differential form in $\mathbb{R}^n$ is simply a differentiable function in $\mathbb{R}^n$.

231 Definition A *k-differential form field in* $\mathbb{R}^n$ is an expression of the form

$$\omega = \sum_{1\leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} a_{j_1 j_2 \ldots j_k} dx_{j_1} \wedge dx_{j_2} \wedge \cdots dx_{j_k},$$

where the $a_{j_1 j_2 \ldots j_k}$ are differentiable functions in $\mathbb{R}^n$.

232 Example

$$g(x, y, z, w) = x + y^2 + z^3 + w^4$$

is a 0-form in $\mathbb{R}^4$.

233 Example An example of a 1-form field in $\mathbb{R}^3$ is

$$\omega = xdx + y^2dy + xyz^3dz.$$

234 Example An example of a 2-form field in $\mathbb{R}^3$ is

$$\omega = x^2dx \wedge dy + y^2dy \wedge dz + dz \wedge dx.$$

235 Example An example of a 3-form field in $\mathbb{R}^3$ is

$$\omega = (x + y + z)dx \wedge dy \wedge dz.$$

We shew now how to multiply differential forms.

236 Example The product of the 1-form fields in $\mathbb{R}^3$

$$\omega_1 = ydx + xdy,$$

$$\omega_2 = -2xdx + 2ydy,$$

is

$$\omega_1 \wedge \omega_2 = (2x^2 + 2y^2)dx \wedge dy.$$

237 Definition Let $f(x_1, x_2, \ldots, x_n)$ be a 0-form in $\mathbb{R}^n$. The *exterior derivative* df of f is

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Furthermore, if

$$\omega = f(x_1, x_2, \ldots, x_n)\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}$$

is a k-form in $\mathbb{R}^n$, the *exterior derivative* $\mathrm{d}\omega$ of ω is the $(k+1)$-form

$$\mathrm{d}\omega = \mathrm{d}f(x_1, x_2, \ldots, x_n) \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}.$$

238 Example If in $\mathbb{R}^2$, $\omega = x^3y^4$, then

$$\mathrm{d}(x^3y^4) = 3x^2y^4\mathrm{d}x + 4x^3y^3\mathrm{d}y.$$

239 Example If in $\mathbb{R}^2$, $\omega = x^2y\mathrm{d}x + x^3y^4\mathrm{d}y$ then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(x^2y\mathrm{d}x + x^3y^4\mathrm{d}y) \\
&= (2xy\mathrm{d}x + x^2\mathrm{d}y) \wedge \mathrm{d}x + (3x^2y^4\mathrm{d}x + 4x^3y^3\mathrm{d}y) \wedge \mathrm{d}y \\
&= x^2\mathrm{d}y \wedge \mathrm{d}x + 3x^2y^4\mathrm{d}x \wedge \mathrm{d}y \\
&= (3x^2y^4 - x^2)\mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

240 Example Consider the change of variables $x = u + v, y = uv$. Then

$$\mathrm{d}x = \mathrm{d}u + \mathrm{d}v,$$

$$\mathrm{d}y = v\mathrm{d}u + u\mathrm{d}v,$$

whence

$$\mathrm{d}x \wedge \mathrm{d}y = (u - v)\mathrm{d}u \wedge \mathrm{d}v.$$

241 Example Consider the transformation of coordinates xyz into uvw coordinates given by

$$u = x + y + z,\ v = \frac{z}{y+z},\ w = \frac{y+z}{x+y+z}.$$

Then

$$\mathrm{d}u = \mathrm{d}x + \mathrm{d}y + \mathrm{d}z,$$

$$\mathrm{d}v = -\frac{z}{(y+z)^2}\mathrm{d}y + \frac{y}{(y+z)^2}\mathrm{d}z,$$

$$\mathrm{d}w = -\frac{y+z}{(x+y+z)^2}\mathrm{d}x + \frac{x}{(x+y+z)^2}\mathrm{d}y + \frac{x}{(x+y+z)^2}\mathrm{d}z.$$

Multiplication gives

$$\begin{aligned}
\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w &= \left(-\frac{zx}{(y+z)^2(x+y+z)^2} - \frac{y(y+z)}{(y+z)^2(x+y+z)^2}\right. \\
&\qquad \left. + \frac{z(y+z)}{(y+z)^2(x+y+z)^2} - \frac{xy}{(y+z)^2(x+y+z)^2}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \frac{z^2 - y^2 - zx - xy}{(y+z)^2(x+y+z)^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

3.2 Zero-Manifolds

242 Definition A 0-*dimensional oriented manifold of* $\mathbb{R}^n$ is simply a point $\mathrm{x} \in \mathbb{R}^n$, with a choice of the $+$ or $-$ sign. A general oriented 0-manifold is a union of oriented points.

243 Definition Let $M = +\{\mathrm{b}\} \cup -\{\mathrm{a}\}$ be an oriented 0-manifold, and let ω be a 0-form. Then

$$\int_M \omega = \omega(\mathrm{b}) - \omega(\mathrm{a}).$$

☞ *$-\mathrm{x}$ has opposite orientation to $+\mathrm{x}$ and*

$$\int_{-\mathrm{x}} \omega = -\int_{+\mathrm{x}} \omega.$$

244 Example Let $M = -\{(1,0,0)\} \cup +\{(1,2,3)\} \cup -\{(0,-2,0)\}^2$ be an oriented 0-manifold, and let $\omega = x + 2y + z^2$. Then

$$\int_M \omega = -\omega((1,0,0)) + \omega(1,2,3) - \omega(0,0,3) = -(1) + (14) - (-4) = 17.$$

3.3 One-Manifolds

245 Definition A 1-*dimensional oriented manifold of* $\mathbb{R}^n$ is simply an oriented smooth curve $\Gamma \in \mathbb{R}^n$, with a choice of a $+$ orientation if the curve traverses in the direction of increasing t, or with a choice of a $-$ sign if the curve traverses in the direction of decreasing t. A general oriented 1-manifold is a union of oriented curves.

☞ *The curve $-\Gamma$ has opposite orientation to Γ and*

$$\int_{-\Gamma} \omega = -\int_{\Gamma} \omega.$$

If $\overrightarrow{\mathrm{f}} : \mathbb{R}^2 \to \mathbb{R}^2$ and if $\mathrm{d}\overrightarrow{\mathrm{r}} = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$, the classical way of writing this is

$$\int_{\Gamma} \overrightarrow{\mathrm{f}} \bullet \mathrm{d}\overrightarrow{\mathrm{r}}.$$

We now turn to the problem of integrating 1-forms.

246 Example Calculate

$$\int_{\Gamma} xy\mathrm{d}x + (x+y)\mathrm{d}y$$

where Γ is the parabola $y = x^2$, $x \in [-1;2]$ oriented in the positive direction.

Solution: ► *We parametrise the curve as $x = t, y = t^2$. Then*

$$xy\mathrm{d}x + (x+y)\mathrm{d}y = t^3\mathrm{d}t + (t+t^2)\mathrm{d}t^2 = (3t^3 + 2t^2)\mathrm{d}t,$$

whence

$$\begin{aligned} \int_{\Gamma} \omega &= \int_{-1}^{2} (3t^3 + 2t^2)\mathrm{d}t \\ &= \left[\frac{2}{3}t^3 + \frac{3}{4}t^4\right]_{-1}^{2} \\ &= \frac{69}{4}. \end{aligned}$$

What would happen if we had given the curve above a different parametrisation? First observe that the curve travels from $(-1,1)$ to $(2,4)$ on the parabola $y = x^2$. These conditions are met with the parametrisation $x = \sqrt{t} - 1, y = (\sqrt{t} - 1)^2$, $t \in [0;9]$. Then

$$\begin{aligned} xy\mathrm{d}x + (x+y)\mathrm{d}y &= (\sqrt{t}-1)^3\mathrm{d}(\sqrt{t}-1) + ((\sqrt{t}-1) + (\sqrt{t}-1)^2)\mathrm{d}(\sqrt{t}-1)^2 \\ &= (3(\sqrt{t}-1)^3 + 2(\sqrt{t}-1)^2)\mathrm{d}(\sqrt{t}-1) \\ &= \frac{1}{2\sqrt{t}}(3(\sqrt{t}-1)^3 + 2(\sqrt{t}-1)^2)\mathrm{d}t, \end{aligned}$$

[2] Do not confuse, say, $-\{(1,0,0)\}$ with $-(1,0,0) = (-1,0,0)$. The first one means that the point $(1,0,0)$ is given negative orientation, the second means that $(-1,0,0)$ is the additive inverse of $(1,0,0)$.

whence

$$\begin{aligned}\int_\Gamma \omega &= \int_0^9 \frac{1}{2\sqrt{t}}(3(\sqrt{t}-1)^3+2(\sqrt{t}-1)^2)\mathrm{d}t\\ &= \left[\frac{3t^2}{4}-\frac{7t^{3/2}}{3}+\frac{5t}{2}-\sqrt{t}\right]_0^9\\ &= \frac{69}{4},\end{aligned}$$

as before.

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <x*y,x+y> ), Path( <t,t^2>, t=-1..2));
```

◀

It turns out that if two different parametrisations of the same curve have the same orientation, then their integrals are equal. Hence, we only need to worry about finding a suitable parametrisation.

247 Example Calculate the line integral

$$\int_\Gamma y\sin x\mathrm{d}x + x\cos y\mathrm{d}y,$$

where Γ is the line segment from $(0,0)$ to $(1,1)$ in the positive direction.

Solution: ▶ *This line has equation $y=x$, so we choose the parametrisation $x=y=t$. The integral is thus*

$$\begin{aligned}\int_\Gamma y\sin x\mathrm{d}x + x\cos y\mathrm{d}y &= \int_0^1 (t\sin t+t\cos t)\mathrm{d}t\\ &= [t(\sin x-\cos t)]_0^1 - \int_0^1(\sin t-\cos t)\mathrm{d}t\\ &= 2\sin 1-1,\end{aligned}$$

upon integrating by parts.

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <y*sin(x),x*cos(y)> ), Line(<0,0>,<1,1>));
```

◀

248 Example Calculate the path integral

$$\int_\Gamma \frac{x+y}{x^2+y^2}\ \mathrm{d}y + \frac{x-y}{x^2+y^2}\ \mathrm{d}x$$

around the closed square $\Gamma = ABCD$ with $A=(1,1)$, $B=(-1,1)$, $C=(-1,-1)$, and $D=(1,-1)$ in the direction $ABCDA$.

Solution: ▶ *On AB, $y=1, \mathrm{d}y=0$, on BC, $x=-1, \mathrm{d}x=0$, on CD, $y=-1, \mathrm{d}y=0$, and on DA, $x=1, \mathrm{d}x=0$. The integral is thus*

$$\begin{aligned}\int_\Gamma\omega &= \int_{AB}\omega+\int_{BC}\omega+\int_{CD}\omega+\int_{DA}\omega\\ &= \int_1^{-1}\frac{x-1}{x^2+1}\,\mathrm{d}x+\int_1^{-1}\frac{y-1}{y^2+1}\,\mathrm{d}y+\int_{-1}^1\frac{x+1}{x^2+1}\,\mathrm{d}x+\int_{-1}^1\frac{y+1}{y^2+1}\,\mathrm{d}y\\ &= 4\int_{-1}^1\frac{1}{x^2+1}\,\mathrm{d}x\\ &= 4\arctan x|_{-1}^1\\ &= 2\pi.\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <(x+y)/(x^2+y^2),(x-y)/(x^2+y^2)> ),
> LineSegments(<1,1>,<-1,1>,<-1,-1>,<1,-1>,<1,1>));
```

◀

☞ *When the integral is along a closed path, like in the preceding example, it is customary to use the symbol* $\oint_\Gamma$ *rather than* $\int_\Gamma$. *The positive direction of integration is that sense that when traversing the path, the area enclosed by the curve is to the left of the curve.*

249 Example Calculate the path integral

$$\oint_\Gamma x^2 \mathrm{d}y + y^2 \mathrm{d}x,$$

where Γ is the ellipse $9x^2 + 4y^2 = 36$ traversed once in the positive sense.

Solution: ▶ *Parametrise the ellipse as* $x = 2\cos t, y = 3\sin t, t \in [0; 2\pi]$. *Observe that when traversing this closed curve, the area of the ellipse is on the left hand side of the path, so this parametrisation traverses the curve in the positive sense. We have*

$$\begin{aligned}\oint_\Gamma \omega &= \int_0^{2\pi} ((4\cos^2 t)(3\cos t) + (9\sin t)(-2\sin t))\mathrm{d}t \\ &= \int_0^{2\pi} (12\cos^3 t - 18\sin^3 t)\mathrm{d}t \\ &= 0.\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <y^2,x^2> ),Ellipse(9*x^2 + 4*y^2 -36));
```

◀

250 Definition Let Γ be a smooth curve. The integral

$$\int_\Gamma f(\mathrm{x})||\mathrm{d}\mathrm{x}||$$

is called the *path integral of* f *along* Γ.

251 Example Find $\int_\Gamma x||\mathrm{d}\mathrm{x}||$ where Γ is the triangle starting at $A : (-1,-1)$ to $B : (2,-2)$, and ending in $C : (1,2)$.

Solution: ▶ *The lines passing through the given points have equations* $L_{AB} : y = \dfrac{-x-4}{3}$, *and* $L_{BC} : y = -4x + 6$. *On* L_{AB}

$$x||\mathrm{d}\mathrm{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = x\sqrt{1 + \left(-\frac{1}{3}\right)^2}\mathrm{d}x = \frac{x\sqrt{10}\mathrm{d}x}{3},$$

and on L_{BC}

$$x||\mathrm{d}\mathrm{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = x(\sqrt{1 + (-4)^2})\mathrm{d}x = x\sqrt{17}\mathrm{d}x.$$

Hence

$$
\begin{aligned}
\int_{\Gamma} x||\mathrm{d}\mathbf{x}|| &= \int_{L_{AB}} x||\mathrm{d}\mathbf{x}|| + \int_{L_{BC}} x||\mathrm{d}\mathbf{x}|| \\
&= \int_{-1}^{2} \frac{x\sqrt{10}\mathrm{d}x}{3} + \int_{2}^{1} x\sqrt{17}\mathrm{d}x \\
&= \frac{\sqrt{10}}{2} - \frac{3\sqrt{17}}{2}.
\end{aligned}
$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> PathInt( x, [x,y]=LineSegments( <-1,-1>, <2,-2>,<1,2> ) );
```

◀

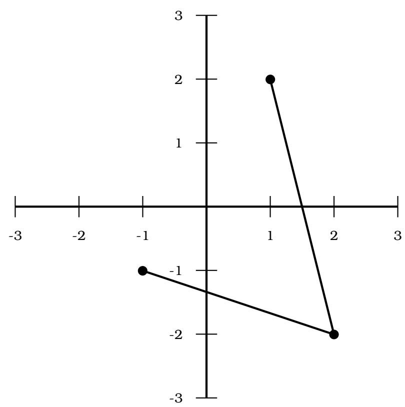

Figure 3.1: Example 251.

Homework

Problem 3.3.1 Consider $\int_C x\mathrm{d}x + y\mathrm{d}y$ and $\int_C xy||\mathrm{d}\mathbf{x}||$.

1. Evaluate $\int_C x\mathrm{d}x + y\mathrm{d}y$ where C is the straight line path that starts at $(-1, 0)$ goes to $(0, 1)$ and ends at $(1, 0)$, by parametrising this path. Calculate also $\int_C xy||\mathrm{d}\mathbf{x}||$ using this parametrisation.
2. Evaluate $\int_C x\mathrm{d}x + y\mathrm{d}y$ where C is the semicircle that starts at $(-1, 0)$ goes to $(0, 1)$ and ends at $(1, 0)$, by parametrising this path. Calculate also $\int_C xy||\mathrm{d}\mathbf{x}||$ using this parametrisation.

Problem 3.3.2 Find $\int_\Gamma x\mathrm{d}x + y\mathrm{d}y$ where Γ is the path shewn in figure 3.2, starting at $O(0, 0)$ going on a straight line to $A\left(4\cos\frac{\pi}{6}, 4\sin\frac{\pi}{6}\right)$ and continuing on an arc of a circle to $B\left(4\cos\frac{\pi}{5}, 4\sin\frac{\pi}{5}\right)$.

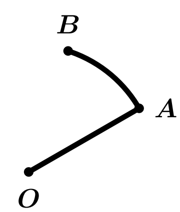

Figure 3.2: Problems 3.3.2 and 3.3.3.

Problem 3.3.3 Find $\int_\Gamma x||\mathrm{d}\mathbf{x}||$ where Γ is the path shewn in figure 3.2.

Problem 3.3.4 Find $\oint_\Gamma z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z$ where Γ is the intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $x + y = 1$, traversed in the positive direction.

3.4 Closed and Exact Forms

252 Lemma (Poincaré Lemma) If ω is a p-differential form of continuously differentiable functions in $\mathbb{R}^n$ then

$$\mathrm{d}(\mathrm{d}\omega) = 0.$$

Proof: *We will prove this by induction on p. For $p = 0$ if*

$$\omega = f(x_1, x_2, \ldots, x_n)$$

then

$$\mathrm{d}\omega = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \mathrm{d}x_k$$

and

$$\begin{aligned} \mathrm{d}(\mathrm{d}\omega) &= \sum_{k=1}^{n} \mathrm{d}\left(\frac{\partial f}{\partial x_k}\right) \wedge \mathrm{d}x_k \\ &= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} \wedge \mathrm{d}x_j\right) \wedge \mathrm{d}x_k \\ &= \sum_{1 \leq j \leq k \leq n}^{n} \left(\frac{\partial^2 f}{\partial x_j \partial x_k} - \frac{\partial^2 f}{\partial x_k \partial x_j}\right) \mathrm{d}x_j \wedge \mathrm{d}x_k \\ &= 0, \end{aligned}$$

since ω is continuously differentiable and so the mixed partial derivatives are equal. Consider now an arbitrary p-form, $p > 0$. Since such a form can be written as

$$\omega = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_p \leq n} a_{j_1 j_2 \ldots j_p} \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p},$$

where the $a_{j_1 j_2 \ldots j_p}$ are continuous differentiable functions in $\mathbb{R}^n$, we have

$$\begin{aligned} \mathrm{d}\omega &= \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_p \leq n} \mathrm{d}a_{j_1 j_2 \ldots j_p} \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p} \\ &= \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_p \leq n} \left(\sum_{i=1}^{n} \frac{\partial a_{j_1 j_2 \ldots j_p}}{\partial x_i} \mathrm{d}x_i\right) \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}, \end{aligned}$$

it is enough to prove that for each summand

$$\mathrm{d}\left(\mathrm{d}a \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) = 0.$$

But

$$\begin{aligned} \mathrm{d}\left(\mathrm{d}a \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) &= \mathrm{dd}a \wedge \left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) \\ &\quad + \mathrm{d}a \wedge \mathrm{d}\left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) \\ &= \mathrm{d}a \wedge \mathrm{d}\left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right), \end{aligned}$$

since $\mathrm{dd}a = 0$ from the case $p = 0$. But an independent induction argument proves that

$$\mathrm{d}\left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) = 0,$$

completing the proof. ❑

253 Definition A differential form ω is said to be *exact* if there is a continuously differentiable function F such that

$$\mathrm{d}F = \omega.$$

254 Example The differential form

$$x\mathrm{d}x + y\mathrm{d}y$$

is exact, since

$$x\mathrm{d}x + y\mathrm{d}y = \mathrm{d}\left(\frac{1}{2}(x^2+y^2)\right).$$

255 Example The differential form

$$y\mathrm{d}x + x\mathrm{d}y$$

is exact, since

$$y\mathrm{d}x + x\mathrm{d}y = \mathrm{d}\left(xy\right).$$

256 Example The differential form

$$\frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y$$

is exact, since

$$\frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y = \mathrm{d}\left(\frac{1}{2}\log_e(x^2+y^2)\right).$$

☞ *Let $\omega = \mathrm{d}F$ be an exact form. By the Poincaré Lemma Theorem 252, $\mathrm{d}\omega = \mathrm{dd}F = 0$. A result of Poincaré says that for certain domains (called* star-shaped domains*) the converse is also true, that is, if $\mathrm{d}\omega = 0$ on a star-shaped domain then ω is exact.*

257 Example Determine whether the differential form

$$\omega = \frac{2x(1-e^y)}{(1+x^2)^2}\mathrm{d}x + \frac{e^y}{1+x^2}\mathrm{d}y$$

is exact.

Solution: ► *Assume there is a function F such that*

$$\mathrm{d}F = \omega.$$

By the Chain Rule

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y.$$

This demands that

$$\frac{\partial F}{\partial x} = \frac{2x(1-e^y)}{(1+x^2)^2},$$

$$\frac{\partial F}{\partial y} = \frac{e^y}{1+x^2}.$$

We have a choice here of integrating either the first, or the second expression. Since integrating the second expression (with respect to y) is easier, we find

$$F(x,y) = \frac{e^y}{1+x^2} + \phi(x),$$

where $\phi(x)$ is a function depending only on x. To find it, we differentiate the obtained expression for F with respect to x and find

$$\frac{\partial F}{\partial x} = -\frac{2xe^y}{(1+x^2)^2} + \phi'(x).$$

Comparing this with our first expression for $\frac{\partial F}{\partial x}$, *we find*

$$\phi'(x) = \frac{2x}{(1+x^2)^2},$$

that is

$$\phi(x) = -\frac{1}{1+x^2} + c,$$

where c *is a constant. We then take*

$$F(x, y) = \frac{e^y - 1}{1+x^2} + c.$$

◀

258 Example Is there a continuously differentiable function such that

$$\mathrm{d}F = \omega = y^2z^3\mathrm{d}x + 2xyz^3\mathrm{d}y + 3xy^2z^2\mathrm{d}z \quad ?$$

Solution: ▶ *We have*

$$\begin{aligned} \mathrm{d}\omega &= (2yz^3\mathrm{d}y + 3y^2z^2\mathrm{d}z) \wedge \mathrm{d}x \\ &\quad +(2yz^3\mathrm{d}x + 2xz^3\mathrm{d}y + 6xyz^2\mathrm{d}z) \wedge \mathrm{d}y \\ &\quad +(3y^2z^2\mathrm{d}x + 6xyz^2\mathrm{d}y + 6xy^2z\mathrm{d}z) \wedge \mathrm{d}z \\ &= 0, \end{aligned}$$

so this form is exact in a star-shaped domain. So put

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z = y^2z^3\mathrm{d}x + 2xyz^3\mathrm{d}y + 3xy^2z^2\mathrm{d}z.$$

Then

$$\begin{aligned} \frac{\partial F}{\partial x} = y^2z^3 &\implies F = xy^2z^3 + a(y, z), \\ \frac{\partial F}{\partial y} = 2xyz^3 &\implies F = xy^2z^3 + b(x, z), \\ \frac{\partial F}{\partial z} = 3xy^2z^2 &\implies F = xy^2z^3 + c(x, y), \end{aligned}$$

Comparing these three expressions for F, *we obtain* $F(x, y, z) = xy^2z^3$. ◀

We have the following equivalent of the Fundamental Theorem of Calculus.

259 Theorem Let $U \subseteq \mathbb{R}^n$ be an open set. Assume $\omega = \mathrm{d}F$ is an exact form, and Γ a path in U with starting point A and endpoint B. Then

$$\int_\Gamma \omega = \int_A^B \mathrm{d}F = F(B) - F(A).$$

In particular, if Γ is a simple closed path, then

$$\oint_\Gamma \omega = 0.$$

260 Example Evaluate the integral

$$\oint_\Gamma \frac{2x}{x^2+y^2}\mathrm{d}x + \frac{2y}{x^2+y^2}\,\mathrm{d}y$$

where Γ is the closed polygon with vertices at $A = (0, 0)$, $B = (5, 0)$, $C = (7, 2)$, $D = (3, 2)$, $E = (1, 1)$, traversed in the order $ABCDEA$.

Solution: ▶ *Observe that*

$$\mathrm{d}\left(\frac{2x}{x^2+y^2}\mathrm{d}x+\frac{2y}{x^2+y^2}\ \mathrm{d}y\right)=-\frac{4xy}{(x^2+y^2)^2}\mathrm{d}y\wedge\mathrm{d}x-\frac{4xy}{(x^2+y^2)^2}\mathrm{d}x\wedge\mathrm{d}y=0,$$

and so the form is exact in a start-shaped domain. By virtue of Theorem 259, the integral is 0. ◀

261 Example Calculate the path integral

$$\oint_{\Gamma}(x^2-y)\mathrm{d}x+(y^2-x)\mathrm{d}y,$$

where Γ is a loop of $x^3+y^3-2xy=0$ traversed once in the positive sense.

Solution: ▶ *Since*

$$\frac{\partial}{\partial y}(x^2-y)=-1=\frac{\partial}{\partial x}(y^2-x),$$

the form is exact, and since this is a closed simple path, the integral is 0. ◀

3.5 Two-Manifolds

262 Definition A 2-*dimensional oriented manifold of* $\mathbb{R}^2$ is simply an open set (region) $D\in\mathbb{R}^2$, where the $+$ orientation is counter-clockwise and the $-$ orientation is clockwise. A general oriented 2-manifold is a union of open sets.

☞ *The region* $-D$ *has opposite orientation to* D *and*

$$\int_{-D}\omega=-\int_D\omega.$$

We will often write

$$\int_D f(x,y)\mathrm{d}A$$

where $\mathrm{d}A$ *denotes the* area element.

☞ *In this section, unless otherwise noticed, we will choose the positive orientation for the regions considered. This corresponds to using the area form* $\mathrm{d}x\mathrm{d}y$.

Let $D\subseteq\mathbb{R}^2$. Given a function $f:D\to\mathbb{R}$, the integral

$$\int_D f\mathrm{d}A$$

is the sum of all the values of f restricted to D. In particular,

$$\int_D \mathrm{d}A$$

is the area of D.

In order to evaluate double integrals, we need the following.

263 Theorem (Fubini's Theorem) Let $D=[a;b]\times[c;d]$, and let $f:A\to\mathbb{R}$ be continuous. Then

$$\int_D f\mathrm{d}A=\int_a^b\left(\int_c^d f(x,y)\mathrm{d}y\right)\mathrm{d}x=\int_c^d\left(\int_a^b f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

Fubini's Theorem allows us to convert the double integral into iterated (single) integrals.

264 Example

$$\begin{aligned}\int_{[0;1]\times[2;3]} xy\mathrm{d}A &= \int_0^1 \left(\int_2^3 xy\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1 \left(\left[\frac{xy^2}{2}\right]_2^3\right)\mathrm{d}x \\ &= \int_0^1 \left(\frac{9x}{2} - 2x\right)\mathrm{d}x \\ &= \left[\frac{5x^2}{4}\right]_0^1 \\ &= \frac{5}{4}.\end{aligned}$$

Notice that if we had integrated first with respect to x we would have obtained the same result:

$$\begin{aligned}\int_2^3 \left(\int_0^1 xy\mathrm{d}x\right)\mathrm{d}y &= \int_2^3 \left(\left[\frac{x^2y}{2}\right]_0^1\right)\mathrm{d}y \\ &= \int_2^3 \left(\frac{y}{2}\right)\mathrm{d}x \\ &= \left[\frac{y^2}{4}\right]_2^3 \\ &= \frac{5}{4}.\end{aligned}$$

Also, this integral is "factorable into x and y pieces" meaning that

$$\begin{aligned}\int_{[0;1]\times[2;3]} xy\mathrm{d}A &= \left(\int_0^1 x\mathrm{d}x\right)\left(\int_2^3 y\mathrm{d}y\right) \\ &= \left(\frac{1}{2}\right)\left(\frac{5}{2}\right) \\ &= \frac{5}{4}\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
>  with(Student[VectorCalculus]):
>  int(x*y, [x,y]=Region(0..1,2..3));
```

265 Example We have

$$\begin{aligned}\int_3^4\int_0^1 (x+2y)(2x+y)\ \mathrm{d}x\mathrm{d}y &= \int_3^4\int_0^1 (2x^2+5xy+2y^2)\ \mathrm{d}x\mathrm{d}y \\ &= \int_3^4 \left(\frac{2}{3}+\frac{5}{2}y+2y^2\right)\ \mathrm{d}y \\ &= \frac{409}{12}.\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
>  with(Student[VectorCalculus]):
>  int((x + 2*y)*(2*x + y), [x,y]=Region(3..4,0..1));
```

In the cases when the domain of integration is not a rectangle, we decompose so that, one variable is kept constant.

266 Example Find $\int_D xy\ \mathrm{d}x\mathrm{d}y$ in the triangle with vertices $A:(-1,-1)$, $B:(2,-2)$, $C:(1,2)$.

Solution: ► *The lines passing through the given points have equations* $L_{AB}: y = \frac{-x-4}{3}$, $L_{BC}: y = -4x+6$, $L_{CA}: y = \frac{3x+1}{2}$. *Now, we draw the region* carefully. *If we integrate first with respect to* y, *we must divide the region as in figure 3.3, because there are two upper lines which the upper value of* y *might be. The lower point of the dashed line is* $(1, -5/3)$. *The integral is thus*

$$\int_{-1}^{1} x \left(\int_{(-x-4)/3}^{(3x+1)/2} y \, dy \right) dx + \int_{1}^{2} x \left(\int_{(-x-4)/3}^{-4x+6} y \, dy \right) dx = -\frac{11}{8}.$$

If we integrate first with respect to x, *we must divide the region as in figure 3.4, because there are two left-most lines which the left value of* x *might be. The right point of the dashed line is* $(7/4, -1)$. *The integral is thus*

$$\int_{-2}^{-1} y \left(\int_{-4-3y}^{(6-y)/4} x \, dx \right) dy + \int_{-1}^{2} y \left(\int_{(2y-1)/3}^{(6-y)/4} x \, dx \right) dy = -\frac{11}{8}.$$

To solve this problem using Maple you may use the code below.

```
>  with(Student[VectorCalculus]):
>  int(x*y, [x,y]=Triangle(<-1,-1>,<2,-2>,<1,2>);
```

◄

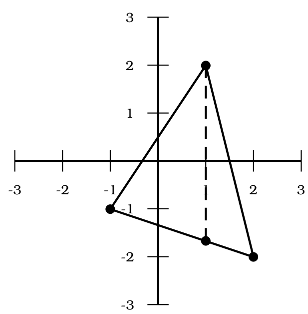

Figure 3.3: Example 266. Integration order $dydx$.

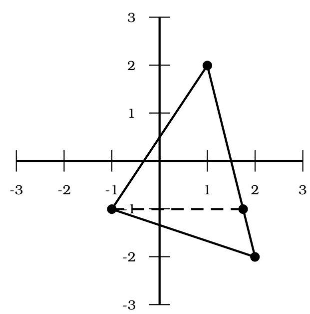

Figure 3.4: Example 266. Integration order $dxdy$.

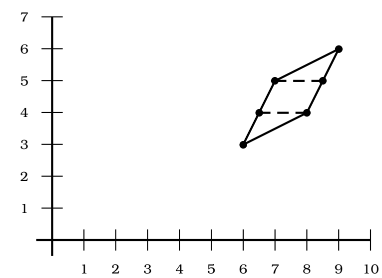

Figure 3.5: Example 267.

267 Example Consider the region inside the parallelogram P with vertices at $A : (6,3)$, $B : (8,4)$, $C : (9,6)$, $D : (7,5)$, as in figure 3.5. Find

$$\int_P xy \; dxdy.$$

Solution: ► *The lines joining the points have equations*

$$L_{AB}: \; y = \frac{x}{2},$$

$$L_{BC}: \; y = 2x - 12,$$

$$L_{CD}: \; y = \frac{x}{2} + \frac{3}{2},$$

$$L_{DA}: \; y = 2x - 9.$$

The integral is thus

$$\int_3^4 \int_{(y+9)/2}^{2y} xy \, dxdy + \int_4^5 \int_{(y+9)/2}^{(y+12)/2} xy \, dxdy + \int_5^6 \int_{2y-3}^{(y+12)/2} xy \, dxdy = \frac{409}{4}.$$

To solve this problem using Maple you may use the code below. Notice that we have split the parallelogram into two triangles.

```
> with(Student[VectorCalculus]):
> int(x*y, [x,y]=Triangle(<6,3>,<8,4>,<7,5>))
> + int(x*y, [x,y]=Triangle(<8,4>,<9,6>,<7,5>));
```

◄

268 Example Find

$$\int_D \frac{y}{x^2+1} dxdy$$

where

$$D = \{(x,y) \in \mathbb{R}^2 | x \geq 0, x^2 + y^2 \leq 1\}.$$

Solution: ► *The integral is* 0. *Observe that if* $(x,y) \in D$ *then* $(x,-y) \in D$. *Also,* $f(x,-y) = -f(x,y)$. ◄

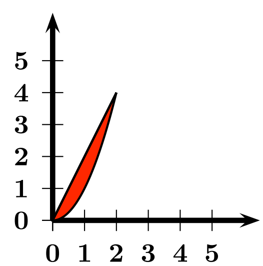

Figure 3.6: Example 269.

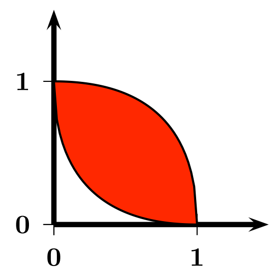

Figure 3.7: Example 270.

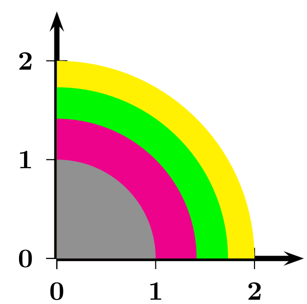

Figure 3.8: Example 271.

269 Example Find

$$\int_0^4 \left(\int_{y/2}^{\sqrt{y}} e^{y/x} \quad dx \right) dy.$$

Solution: ► *We have*

$$0 \leq y \leq 4, \quad \frac{y}{2} \leq x \leq \sqrt{y} \implies 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x.$$

We then have

$$\begin{aligned}
\int_0^4 \left(\int_{y/2}^{\sqrt{y}} e^{y/x} dx \right) dy &= \int_0^2 \left(\int_{x^2}^{2x} e^{y/x} dy \right) dx \\
&= \int_0^2 \left(x e^{y/x} \Big|_{x^2}^{2x} \right) dx \\
&= \int_0^2 (xe^2 - xe^x) \, dx \\
&= 2e^2 - (2e^2 - e^2 + 1) \\
&= e^2 - 1
\end{aligned}$$

◄

270 Example Find the area of the region

$$R = \{(x, y) \in \mathbb{R}^2 : \sqrt{x} + \sqrt{y} \geq 1, \quad \sqrt{1-x} + \sqrt{1-y} \geq 1\}.$$

Solution: ► *The area is given by*

$$\begin{aligned}\int_D \mathrm{d}A &= \int_0^1 \left(\int_{(1-\sqrt{x})^2}^{1-(1-\sqrt{1-x})^2} \mathrm{d}y \right) \mathrm{d}x \\ &= 2\int_0^1 (\sqrt{1-x} + \sqrt{x} - 1)\mathrm{d}x \\ &= \frac{2}{3}.\end{aligned}$$

◄

271 Example Evaluate $\int_R \llbracket x^2 + y^2 \rrbracket \mathrm{d}A$, where R is the rectangle $[0; \sqrt{2}] \times [0; \sqrt{2}]$.

Solution: ► *The function* $(x, y) \mapsto \llbracket x^2 + y^2 \rrbracket$ *jumps every time* $x^2 + y^2$ *is an integer. For* $(x, y) \in R$, *we have* $0 \leq x^2 + y^2 \leq (\sqrt{2})^2 + (\sqrt{2})^2 = 4$. *Thus we decompose* R *as the union of the*

$$R_k = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, k \leq x^2 + y^2 < k + 1\}, \qquad k \in \{1, 2, 3\}.$$

$$\begin{aligned}\int_R \llbracket x^2 + y^2 \rrbracket \mathrm{d}A &= \sum_{1 \leq k \leq 3} \int_{R_k} \llbracket x^2 + y^2 \rrbracket \mathrm{d}A \\ &= \iint_{1 \leq x^2+y^2 < 2, x \geq 0, y \geq 0} 1\mathrm{d}A + \iint_{2 \leq x^2+y^2 < 3, x \geq 0, y \geq 0} 2\mathrm{d}A + \iint_{3 \leq x^2+y^2 < 4, x \geq 0, y \geq 0} 3\mathrm{d}A.\end{aligned}$$

Now the integrals can be computed by realising that they are areas of quarter annuli, and so,

$$\iint_{k \leq x^2+y^2 < k+1, x \geq 0, y \geq 0} k\mathrm{d}A = k \cdot \frac{1}{4} \cdot \pi(k + 1 - k) = \frac{\pi k}{4}.$$

Hence

$$\int_R \llbracket x^2 + y^2 \rrbracket \mathrm{d}A = \frac{\pi}{4}(1 + 2 + 3) = \frac{3\pi}{2}.$$

◄

Homework

Problem 3.5.1 Evaluate the iterated integral $\int_1^3 \int_0^x \frac{1}{x} \mathrm{d}y\mathrm{d}x$.

Problem 3.5.2 Let S be the interior and boundary of the triangle with vertices $(0, 0)$, $(2, 1)$, and $(2, 0)$. Find $\int_S y\mathrm{d}A$.

Problem 3.5.3 Let

$$S = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0, \ 1 \leq x^2 + y^2 \leq 4\}.$$

Find $\int_S x^2 \mathrm{d}A$.

Problem 3.5.4 Find

$$\int_D xy\mathrm{d}x\mathrm{d}y$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | y \geq x^2, x \geq y^2\}.$$

Problem 3.5.5 Find

$$\int_D (x + y)(\sin x)(\sin y)\mathrm{d}A$$

where $D = [0; \pi]^2$.

Problem 3.5.6 Find $\int_0^1 \int_0^1 \min(x^2, y^2)\mathrm{d}x\mathrm{d}y$.

Problem 3.5.7 Find $\int_D xy\mathrm{d}x\mathrm{d}y$ where

$D = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, 9 < x^2+y^2 < 16, 1 < x^2-y^2 < 16\}$.

Problem 3.5.8 Evaluate $\int_{\mathcal{R}} x\mathrm{d}A$ where $\mathcal{R}$ is the (unoriented) circular segment in figure 3.9, which is created by the intersection of regions

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 16\}$$

and

$$\left\{(x, y) \in \mathbb{R}^2 : y \geq -\frac{\sqrt{3}}{3}x + 4\right\}.$$

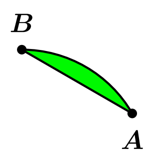

Figure 3.9: Problem 3.5.8.

Problem 3.5.9 Find $\int_0^1 \int_y^1 2e^{x^2}\mathrm{d}x\mathrm{d}y$

Problem 3.5.10 Evaluate $\int_{[0;1]^2} \min(x, y^2)\mathrm{d}A$.

Problem 3.5.11 Find $\int_{\mathcal{R}} xy\mathrm{d}A$, where $\mathcal{R}$ is the (unoriented) $\triangle OAB$ in figure 3.10 with $O(0, 0)$, $A(3, 1)$, and $B(4, 4)$.

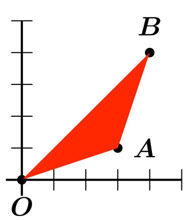

Figure 3.10: Problem 3.5.11.

Problem 3.5.12 Find

$$\int_D \log_e(1 + x + y)\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq 1\}.$$

Problem 3.5.13 Evaluate $\int_{[0;2]^2} \llbracket x + y^2 \rrbracket \mathrm{d}A$.

Problem 3.5.14 Evaluate $\int_R \llbracket x+y \rrbracket \mathrm{d}A$, where R is the rectangle $[0\,;1] \times [0\,;2]$.

Problem 3.5.15 Evaluate $\int_R x\mathrm{d}A$ where R is the quarter annulus in figure 3.11, which formed by the the area between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the first quadrant.

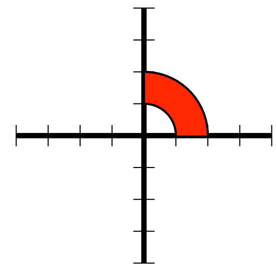

Figure 3.11: Problem 3.5.15.

Problem 3.5.16 Evaluate $\int_R x\mathrm{d}A$ where R is the E-shaped figure in figure 3.12.

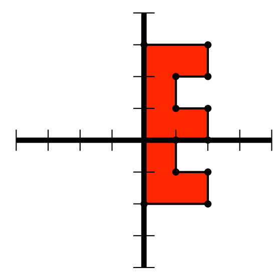

Figure 3.12: Problem 3.5.16.

Problem 3.5.17 Evaluate $\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos y}{y}\mathrm{d}y\mathrm{d}x$.

Problem 3.5.18 Find

$$\int_1^2 \left(\int_{\sqrt{x}}^x \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\,\mathrm{d}x + \int_2^4 \left(\int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\,\mathrm{d}x.$$

Problem 3.5.19 Find

$$\int_D 2x(x^2 + y^2)\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 + x^2 - y^2 \leq 1\}.$$

Problem 3.5.20 Find the area bounded by the ellipses $x^2 + \frac{y^2}{4} = 1$ and $\frac{x^2}{4} + y^2 = 1$, as in figure 3.13.

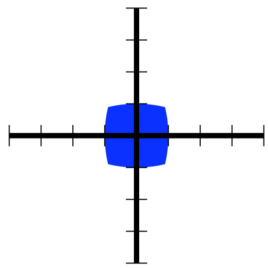

Figure 3.13: Problem 3.5.20.

Problem 3.5.21 Find

$$\int_D xy\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, xy + y + x \leq 1\}.$$

Problem 3.5.22 Find

$$\int_D \log_e(1 + x^2 + y)\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y \leq 1\}.$$

Problem 3.5.23 Evaluate $\int_R x\mathrm{d}A$, where R is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2y$, as shewn in figure 3.14.

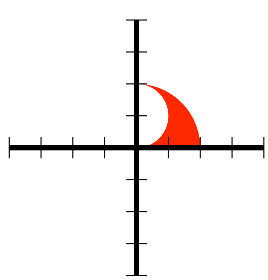

Figure 3.14: Problem 3.5.23.

Problem 3.5.24 Evaluate $\int_0^1 \int_{\sqrt{x}}^1 \frac{e^{x/y}}{y}\mathrm{d}y\mathrm{d}x$.

Problem 3.5.25 Find

$$\int_D |x - y|\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}.$$

Problem 3.5.26 Find $\int_D (2x + 3y + 1)\ \mathrm{d}A$, where D is the triangle with vertices at $A(-1, -1)$, $B(2, -4)$, and $C(1, 3)$.

Problem 3.5.27 Let $f : [0; 1] \to]0; +\infty]$ be a decreasing function. Prove that

$$\frac{\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 xf(x)\mathrm{d}x} \leq \frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}.$$

Problem 3.5.28 Find

$$\int_D (xy(x + y))\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq 1\}.$$

Problem 3.5.29 Let $f, g : [0; 1] \to [0; 1]$ be continuous, with f increasing. Prove that

$$\int_0^1 (f \circ g)(x)\mathrm{d}x \leq \int_0^1 f(x)\mathrm{d}x + \int_0^1 g(x)\mathrm{d}x.$$

Problem 3.5.30 Compute $\int_S (xy + y^2)\mathrm{d}A$ where

$$S = \{(x, y) \in \mathbb{R}^2 : |x|^{1/2} + |y|^{1/2} \leq 1\}.$$

Problem 3.5.31 Evaluate

$$\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)}\ \mathrm{d}y\mathrm{d}x,$$

where a and b are positive.

Problem 3.5.32 Find $\int_D \sqrt{xy}\ \mathrm{d}A$, where

$$D = \{(x, y) \in \mathbb{R}^2 : y \geq 0, (x + y)^2 \leq 2x\}.$$

Problem 3.5.33 A rectangle R on the plane is the disjoint union $R = \cup_{k=1}^N R_k$ of rectangles R_k. It is known that at least one side of each of the rectangles R_k is an integer. Shew that at least one side of R is an integer.

Problem 3.5.34 Evaluate

$$\int_0^1\int_0^1\cdots\int_0^1 (x_1x_2\cdots x_n)\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n.$$

Problem 3.5.35 Evaluate

$$\int_0^1\int_0^1\cdots\int_0^1 (x_1+x_2+\cdots+x_n)\,\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n.$$

Problem 3.5.36 Let I be the rectangle $[1;2]\times[1;2]$ and let f,g be continuous functions $f,g:[1;2]\to[1;2]$ such that $f(x)\leq g(x)$. Demonstrate that

$$\int_I (g(y)-f(x))\,\mathrm{d}x\mathrm{d}y\geq 0.$$

Problem 3.5.37 Find $\int_0^1\int_0^1 x^y\mathrm{d}x\mathrm{d}y$. Then demonstrate that $\int_0^1 \frac{x-1}{\log x}\mathrm{d}x=\log 2$.

Problem 3.5.38 Evaluate

$$\int_0^4\int_0^{\sqrt{4-y}}\sqrt{12x-x^3}\mathrm{d}x\mathrm{d}y.$$

Problem 3.5.39 Evaluate $\int_0^2\int_y^2 y\sqrt{1+x^3}\mathrm{d}x\mathrm{d}y$.

Problem 3.5.40 Evaluate $\int_0^1\int_y^1 \frac{xy}{\sqrt{1+x^4}}\mathrm{d}x\mathrm{d}y$.

Problem 3.5.41 Find

$$\int_D \frac{1}{(x+y)^4}\mathrm{d}A$$

where

$$D=\{(x,y)\in\mathbb{R}^2|x\geq 1, y\geq 1, x+y\leq 4\}.$$

Problem 3.5.42 Prove that

$$\int_0^{+\infty}\int_{2x}^{+\infty}\frac{xe^{-y}\sin y}{y^2}\mathrm{d}y\mathrm{d}x=\frac{1}{16}.$$

Problem 3.5.43 Prove that

$$\int_0^1\int_{y^2}^{y}\frac{y}{x\sqrt{x^2+y^2}}\mathrm{d}x\mathrm{d}y=\log(1+\sqrt{2}).$$

Problem 3.5.44 Prove that

$$\int_0^1\int_0^1\frac{x-y}{(x+y)^3}\mathrm{d}y\mathrm{d}x=\frac{1}{2}=-\int_0^1\int_0^1\frac{x-y}{(x+y)^3}\mathrm{d}x\mathrm{d}y.$$

Is this a contradiction to Fubini's Theorem?

Problem 3.5.45 Find

$$\int_D x\mathrm{d}A$$

where

$$D=\{(x,y)\in\mathbb{R}^2|y\geq 0, x-y+1\geq 0, x+2y-4\leq 0\}.$$

Problem 3.5.46 Evaluate

$$\lim_{n\to+\infty}\int_0^1\int_0^1\cdots\int_0^1\cos^2\left(\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right)\mathrm{d}x_1\,\mathrm{d}x_2\ldots\,\mathrm{d}x$$

Problem 3.5.47 Let f,g be continuous functions in the interval $[a;b]$. Prove that

$$\frac{1}{2}\int_a^b\left(\int_a^b \det\begin{bmatrix}f(x) & g(x)\\ f(y) & g(y)\end{bmatrix}^2\mathrm{d}x\right)\mathrm{d}y$$

equals

$$\left(\int_a^b (f(x))^2\mathrm{d}x\right)\left(\int_a^b (g(x))^2\mathrm{d}x\right)-\left(\int_a^b (f(x)g(x))\mathrm{d}x\right)^2.$$

This is an integral analogue of Lagrange's Identity. Deduce Cauchy's Inequality for integrals,

$$\left(\int_a^b (f(x)g(x))\mathrm{d}x\right)^2\leq\left(\int_a^b (f(x))^2\mathrm{d}x\right)\left(\int_a^b (g(x))^2\mathrm{d}x\right).$$

Problem 3.5.48 Let $a\in\mathbb{R}, n\in\mathbb{N}$, $a>0$, $n>0$. Let $f:[0;a]\to\mathbb{R}$ be continuous. Prove that

$$\int_0^a\int_0^{x_1}\cdots\int_0^{x_{n-2}}\int_0^{x_{n-1}}(f(x_1)f(x_2)\cdots f(x_n))\mathrm{d}x_n\mathrm{d}x_{n-1}\ldots\mathrm{d}x_2\mathrm{d}$$

equals

$$\frac{1}{n!}\left(\int_0^a f(x)\mathrm{d}x\right)^n.$$

3.6 Change of Variables

We now perform a multidimensional analogue of the change of variables theorem in one variable.

272 Theorem Let $(D,\Delta)\in(\mathbb{R}^n)^2$ be open, bounded sets in $\mathbb{R}^n$ with volume and let $g:\Delta\to D$ be a continuously differentiable bijective mapping such that $\det g'(u)\neq 0$, and both $|\det g'(u)|$, $\dfrac{1}{|\det g'(u)|}$

are bounded on Δ. For $f : D \to \mathbb{R}$ bounded and integrable, $f \circ g|\det g'(u)|$ is integrable on Δ and

$$\int \cdots \int_D f = \int \cdots \int_\Delta (f \circ g)|\det g'(u)|,$$

that is

$$\int \cdots \int_D f(x_1, x_2, \ldots, x_n)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$$
$$= \int \cdots \int_\Delta f(g(u_1, u_2, \ldots, u_n))|\det g'(u)|\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \ldots \wedge \mathrm{d}u_n.$$

One normally chooses changes of variables that map into rectangular regions, or that simplify the integrand. Let us start with a rather trivial example.

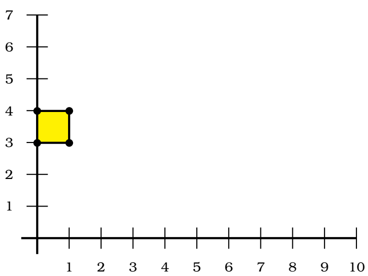

Figure 3.15: Example 273. xy-plane.

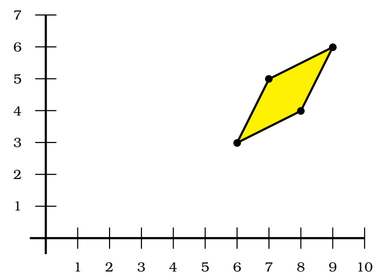

Figure 3.16: Example 273. uv-plane.

273 Example Evaluate the integral

$$\int_3^4 \int_0^1 (x + 2y)(2x + y)\mathrm{d}x\mathrm{d}y.$$

Solution: ► *Observe that we have already computed this integral in example 265. Put*

$$u = x + 2y \implies \mathrm{d}u = \mathrm{d}x + 2\mathrm{d}y,$$

$$v = 2x + y \implies \mathrm{d}v = 2\mathrm{d}x + \mathrm{d}y,$$

giving

$$\mathrm{d}u \wedge \mathrm{d}v = -3\mathrm{d}x \wedge \mathrm{d}y.$$

Now,

$$(u, v) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is a linear transformation, and hence it maps quadrilaterals into quadrilaterals. The corners of the rectangle in the area of integration in the xy-plane are $(0, 3)$, $(1, 3)$, $(1, 4)$, and $(0, 4)$, (traversed counter-clockwise) and they map into $(6, 3)$, $(7, 5)$, $(9, 6)$, and $(8, 4)$, respectively, in the uv-plane (see figure 3.16). The form $\mathrm{d}x \wedge \mathrm{d}y$ has opposite orientation to $\mathrm{d}u \wedge \mathrm{d}v$ so we use

$$\mathrm{d}v \wedge \mathrm{d}u = 3\mathrm{d}x \wedge \mathrm{d}y$$

instead. The integral sought is

$$\frac{1}{3}\int_P uv\ \mathrm{d}v\mathrm{d}u = \frac{409}{12},$$

from example 267. ◀

274 Example The integral

$$\int_{[0;1]^2}(x^4 - y^4)\mathrm{d}A = \int_0^1\left(\frac{1}{5} - y^4\right)\mathrm{d}y = 0.$$

Evaluate it using the change of variables $u = x^2 - y^2, v = 2xy$.

Solution: ► *First we find*

$$\mathrm{d}u = 2x\mathrm{d}x - 2y\mathrm{d}y,$$

$$\mathrm{d}v = 2y\mathrm{d}x + 2x\mathrm{d}y,$$

and so

$$\mathrm{d}u \wedge \mathrm{d}v = (4x^2 + 4y^2)\mathrm{d}x \wedge \mathrm{d}y.$$

We now determine the region Δ *into which the square* $D = [0;1]^2$ *is mapped. We use the fact that boundaries will be mapped into boundaries. Put*

$$AB = \{(x,0) : 0 \leq x \leq 1\},$$

$$BC = \{(1,y) : 0 \leq y \leq 1\},$$

$$CD = \{(1-x,1) : 0 \leq x \leq 1\},$$

$$DA = \{(0,1-y) : 0 \leq y \leq 1\}.$$

On AB *we have* $u = x, v = 0$. *Since* $0 \leq x \leq 1$, AB *is thus mapped into the line segment* $0 \leq u \leq 1, v = 0$.

On BC *we have* $u = 1 - y^2, v = 2y$. *Thus* $u = 1 - \frac{v^2}{4}$. *Hence* BC *is mapped to the portion of the parabola* $u = 1 - \frac{v^2}{4}, 0 \leq v \leq 2$.

On CD *we have* $u = (1-x)^2 - 1, v = 2(1-x)$. *This means that* $u = \frac{v^2}{4} - 1, 0 \leq v \leq 2$.

Finally, on DA, *we have* $u = -(1-y)^2, v = 0$. *Since* $0 \leq y \leq 1$, DA *is mapped into the line segment* $-1 \leq u \leq 0, v = 0$. *The region* Δ *is thus the area in the* uv *plane enclosed by the parabolas* $u \leq \frac{v^2}{4} - 1, u \leq 1 - \frac{v^2}{4}$ *with* $-1 \leq u \leq 1, 0 \leq v \leq 2$.

We deduce that

$$\begin{aligned}\int_{[0;1]^2}(x^4 - y^4)\mathrm{d}A &= \int_\Delta (x^4 - y^4)\frac{1}{4(x^2+y^2)}\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{4}\int_\Delta (x^2 - y^2)\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{4}\int_\Delta u\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{4}\int_0^2\left(\int_{v^2/4-1}^{1-v^2/4} u\mathrm{d}u\right)\mathrm{d}v \\ &= 0,\end{aligned}$$

as before. ◀

275 Example Find

$$\int_D e^{(x^3+y^3)/xy} \quad \mathrm{d}A$$

where

$$D = \{(x,y) \in \mathbb{R}^2 | y^2 - 2px \leq 0, x^2 - 2py \leq 0, p \in]0; +\infty[\text{ fixed}\},$$

using the change of variables $x = u^2 v,\ y = uv^2$.

Solution: ► *We have*

$$\begin{aligned} \mathrm{d}x &= 2uv\mathrm{d}u + u^2\mathrm{d}v, \\ \mathrm{d}y &= v^2\mathrm{d}u + 2uv\mathrm{d}v, \\ \mathrm{d}x \wedge \mathrm{d}y &= 3u^2v^2\mathrm{d}u \wedge \mathrm{d}v. \end{aligned}$$

The region transforms into

$$\Delta = \{(u,v) \in \mathbb{R}^2 | 0 \leq u \leq (2p)^{1/3},\ 0 \leq v \leq (2p)^{1/3}\}.$$

The integral becomes

$$\begin{aligned} \int_D f(x,y)\mathrm{d}x\mathrm{d}y &= \int_\Delta \exp\left(\frac{u^6v^3 + u^3v^6}{u^3v^3}\right)(3u^2v^2)\ \mathrm{d}u\mathrm{d}v \\ &= 3\int_\Delta e^{u^3}e^{v^3}u^2v^2\ \mathrm{d}u\mathrm{d}v \\ &= \frac{1}{3}\left(\int_0^{(2p)^{1/3}} 3u^2e^{u^3}\ \mathrm{d}u\right)^2 \\ &= \frac{1}{3}(e^{2p}-1)^2. \end{aligned}$$

As an exercise, you may try the (more natural) substitution $x^3 = u^2v, y^3 = v^2u$ and verify that the same result is obtained. ◄

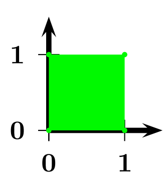

Figure 3.17: Example 276. xy-plane.

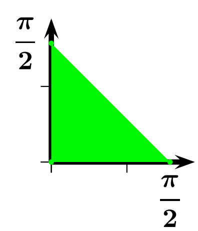

Figure 3.18: Example 276. uv-plane.

276 Example In this problem we will follow an argument of Calabi, Beukers, and Kock to prove that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

1. Prove that if $S = \sum_{n=1}^{+\infty} \frac{1}{n^2}$, then $\frac{3}{4}S = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}$.
2. Prove that $\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = \int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-x^2y^2}$.

3. Use the change of variables $x = \dfrac{\sin u}{\cos v}$, $y = \dfrac{\sin v}{\cos u}$ in order to evaluate $\displaystyle\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1 - x^2y^2}$.

Solution: ►

1. *Observe that the sum of the even terms is*

$$\sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{1}{4} S,$$

a quarter of the sum, hence the sum of the odd terms must be three quarters of the sum, $\dfrac{3}{4} S$.

2. *Observe that*

$$\frac{1}{2n-1} = \int_0^1 x^{2n-2}\mathrm{d}x \implies \left(\frac{1}{2n-1}\right)^2 = \left(\int_0^1 x^{2n-2}\mathrm{d}x\right)\left(\int_0^1 y^{2n-2}\mathrm{d}y\right) = \int_0^1 \int_0^1 (xy)^{2n-2}\mathrm{d}x\mathrm{d}y.$$

Thus

$$\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{+\infty} \int_0^1 \int_0^1 (xy)^{2n-2}\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 \sum_{n=1}^{+\infty} (xy)^{2n-2}\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1 - x^2y^2},$$

as claimed.[3]

3. *If* $x = \dfrac{\sin u}{\cos v}$, $y = \dfrac{\sin v}{\cos u}$, *then*

$$\mathrm{d}x = (\cos u)(\sec v)\mathrm{d}u + (\sin u)(\sec v)(\tan v)\mathrm{d}v, \quad \mathrm{d}y = (\sec u)(\tan u)(\sin v)\mathrm{d}u + (\sec u)(\cos v)\mathrm{d}v,$$

from where

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}u \wedge \mathrm{d}v - (\tan^2 u)(\tan^2 v)\mathrm{d}u \wedge \mathrm{d}v = \left(1 - (\tan^2 u)(\tan^2 v)\right)\mathrm{d}u \wedge \mathrm{d}v.$$

Also,

$$1 - x^2y^2 = 1 - \frac{\sin^2 v}{\cos^2 v} \cdot \frac{\sin^2 v}{\cos^2 u} = 1 - (\tan^2 u)(\tan^2 v).$$

This gives

$$\frac{\mathrm{d}x\mathrm{d}y}{1 - x^2y^2} = \mathrm{d}u\mathrm{d}v.$$

We now have to determine the region that the transformation $x = \dfrac{\sin u}{\cos v}$, $y = \dfrac{\sin v}{\cos u}$ *forms in the* uv*-plane. Observe that*

$$u = \arctan x\sqrt{\frac{1 - y^2}{1 - x^2}}, \qquad v = \arctan y\sqrt{\frac{1 - x^2}{1 - y^2}}.$$

This means that the square in the xy*-plane in figure 3.17 is transformed into the triangle in the* uv*-plane in figure 3.18.*
We deduce,

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1 - x^2y^2} = \int_0^{\pi/2} \int_0^{\pi/2 - v} \mathrm{d}u\mathrm{d}v = \int_0^{\pi/2} (\pi/2 - v)\,\mathrm{d}v = \left(\frac{\pi}{2}v - \frac{v^2}{2}\right)\Big|_0^{\pi/2} = \frac{\pi^2}{4} - \frac{\pi^2}{8} = \frac{\pi^2}{8}.$$

Finally,

$$\frac{3}{4}S = \frac{\pi^2}{8} \implies S = \frac{\pi^2}{6}.$$

◄

[3] This exchange of integral and sum needs justification. We will accept it for our purposes.

Homework

Problem 3.6.1 Let $D' = \{(u, v) \in \mathbb{R}^2 : u \leq 1, -u \leq v \leq u\}$. Consider

$$\Phi : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (u, v) & \mapsto & \left(\dfrac{u+v}{2}, \dfrac{u-v}{2}\right) \end{array}.$$

❶ Find the image of Φ on D', that is, find $D = \Phi(D')$.

❷ Find

$$\int_D (x+y)^2 e^{x^2-y^2} \mathrm{d}A.$$

Problem 3.6.2 Using the change of variables $x = u^2 - v^2$, $y = 2uv$, $u \geq 0$, $v \geq 0$, evaluate $\int_R \sqrt{x^2+y^2} \mathrm{d}A$, where

$$R = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 0 \leq y \leq 2\sqrt{1-|x|}\}$$

Problem 3.6.3 Using the change of variables $u = x - y$ and $v = x + y$, evaluate $\int_R \dfrac{x-y}{x+y} \mathrm{d}A$, where R is the square with vertices at $(0, 2)$, $(1, 1)$, $(2, 2)$, $(1, 3)$.

Problem 3.6.4 Find $\displaystyle\int_D f(x, y) \mathrm{d}A$ where

$$D = \{(x, y) \in \mathbb{R}^2 | a \leq xy \leq b, y \geq x \geq 0, y^2 - x^2 \leq 1, (a, b) \in \mathbb{R}^2, 0 < a < b\}$$

and $f(x, y) = y^4 - x^4$ by using the change of variables $u = xy, v = y^2 - x^2$.

Problem 3.6.5 Use the following steps (due to Tom Apostol) in order to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

❶ Use the series expansion

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \quad |t| < 1,$$

in order to prove (formally) that

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

❷ Use the change of variables $u = x + y, v = x - y$ to shew that

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} = 2\int_0^1 \left(\int_{-u}^{u} \frac{\mathrm{d}v}{4-u^2+v^2}\right) \mathrm{d}u + 2\int_1^2 \left(\int_{u-2}^{2-u} \frac{\mathrm{d}v}{4-u^2+v^2}\right) \mathrm{d}u.$$

❸ Shew that the above integral reduces to

$$2\int_0^1 \frac{2}{\sqrt{4-u^2}} \arctan \frac{u}{\sqrt{4-u^2}} \mathrm{d}u + 2\int_1^2 \frac{2}{\sqrt{4-u^2}} \arctan \frac{2-u}{\sqrt{4-u^2}} \mathrm{d}u.$$

❹ Finally, prove that the above integral is $\dfrac{\pi^2}{6}$ by using the substitution $\theta = \arcsin \dfrac{u}{2}$.

3.7 Change to Polar Coordinates

One of the most common changes of variable is the passage to polar coordinates where

$$x = \rho\cos\theta \implies \mathrm{d}x = \cos\theta \mathrm{d}\rho - \rho\sin\theta \mathrm{d}\theta,$$
$$y = \rho\sin\theta \implies \mathrm{d}y = \sin\theta \mathrm{d}\rho + \rho\cos\theta \mathrm{d}\theta,$$

whence

$$\mathrm{d}x \wedge \mathrm{d}y = (\rho\cos^2\theta + \rho\sin^2\theta)\mathrm{d}\rho \wedge \mathrm{d}\theta = \rho \mathrm{d}\rho \wedge \mathrm{d}\theta.$$

277 Example Find

$$\int_D xy\sqrt{x^2+y^2}\mathrm{d}A$$

where

$$D = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, y \leq x, x^2+y^2 \leq 1\}.$$

Solution: ► *We use polar coordinates. The region D transforms into the region*

$$\Delta = [0;1] \times \left[0;\frac{\pi}{4}\right].$$

Therefore the integral becomes

$$\begin{aligned}\int_\Delta \rho^4\cos\theta\sin\theta \,\mathrm{d}\rho\mathrm{d}\theta &= \left(\int_0^{\pi/4}\cos\theta\sin\theta \,\mathrm{d}\theta\right)\left(\int_0^1 \rho^4 \,\mathrm{d}\rho\right) \\ &= \frac{1}{20}.\end{aligned}$$

◄

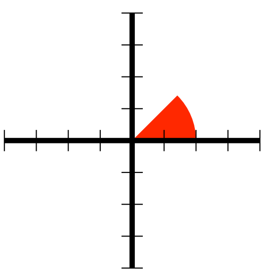

Figure 3.19: Example 277.

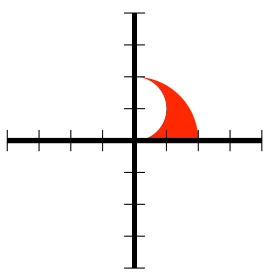

Figure 3.20: Example 278.

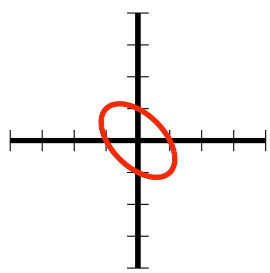

Figure 3.21: Example 279.

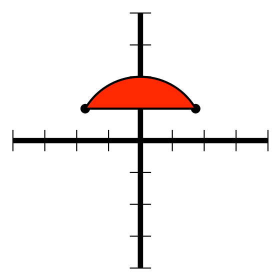

Figure 3.22: Example 280.

278 Example Evaluate $\int_R x\mathrm{d}A$, where R is the region bounded by the circles $x^2+y^2 = 4$ and $x^2+y^2 = 2y$.

Solution: ► *Observe that this is problem 3.5.23. Since $x^2+y^2 = r^2$, the radius sweeps from $r^2 = 2r\sin\theta$ to $r^2 = 4$, that is, from $2\sin\theta$ to 2. The angle clearly sweeps from 0 to $\frac{\pi}{2}$. Thus the integral becomes*

$$\begin{aligned}\int_R x\mathrm{d}A &= \int_0^{\pi/2}\int_2^{2\sin\theta} r^2\cos\theta \mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{3}\int_0^{\pi/2}(8\cos\theta - 8\cos\theta\sin^3\theta)\mathrm{d}\theta \\ &= 2.\end{aligned}$$

◀

279 Example Find $\int_D e^{-x^2-xy-y^2}\,\mathrm{d}A$, where

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+xy+y^2 \leq 1\}.$$

Solution: ► *Completing squares*

$$x^2+xy+y^2 = \left(x+\frac{y}{2}\right)^2 + \left(\frac{\sqrt{3}y}{2}\right)^2.$$

Put $U = x+\frac{y}{2}$, $V = \frac{\sqrt{3}y}{2}$. *The integral becomes*

$$\int_{\{x^2+xy+y^2\leq 1\}} e^{-x^2-xy-y^2}\mathrm{d}x\mathrm{d}y = \frac{2}{\sqrt{3}}\int_{\{U^2+V^2\leq 1\}} e^{-(U^2+V^2)}\mathrm{d}U\mathrm{d}V.$$

Passing to polar coordinates, the above equals

$$\frac{2}{\sqrt{3}}\int_0^{2\pi}\int_0^1 \rho e^{-\rho^2}\mathrm{d}\rho\mathrm{d}\theta = \frac{2\pi}{\sqrt{3}}(1-e^{-1}).$$

◀

280 Example Evaluate $\int_{\mathcal{R}} \frac{1}{(x^2+y^2)^{3/2}}\mathrm{d}A$ over the region $\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2 \leq 4, y \geq 1\right\}$ (figure 3.22).

Solution: ► *The radius sweeps from* $r = \frac{1}{\sin\theta}$ *to* $r = 2$. *The desired integral is*

$$\begin{aligned}\int_{\mathcal{R}} \frac{1}{(x^2+y^2)^{3/2}}\mathrm{d}A &= \int_{\pi/6}^{5\pi/6}\int_{\csc\theta}^{2}\frac{1}{r^2}\mathrm{d}r\mathrm{d}\theta \\ &= \int_{\pi/6}^{5\pi/6}\left(\sin\theta - \frac{1}{2}\right)\mathrm{d}\theta \\ &= \sqrt{3}-\frac{\pi}{3}.\end{aligned}$$

◀

281 Example Evaluate $\int_R (x^3+y^3)\mathrm{d}A$ where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and the first quadrant, $a > 0$ and $b > 0$.

Solution: ► *Put* $x = ar\cos\theta$, $y = br\sin\theta$. *Then*

$$x = ar\cos\theta \implies \mathrm{d}x = a\cos\theta\mathrm{d}r - ar\sin\theta\mathrm{d}\theta,$$

$$y = br\sin\theta \implies \mathrm{d}y = b\sin\theta\mathrm{d}r + br\cos\theta\mathrm{d}\theta,$$

whence

$$\mathrm{d}x\wedge\mathrm{d}y = (abr\cos^2\theta + abr\sin^2\theta)\mathrm{d}r\wedge\mathrm{d}\theta = abr\mathrm{d}r\wedge\mathrm{d}\theta.$$

Observe that on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \implies \frac{a^2r^2\cos^2\theta}{a^2}+\frac{b^2r^2\sin^2\theta}{b^2} = 1 \implies r = 1.$$

Thus the required integral is

$$\begin{aligned}\int_R (x^3+y^3)\mathrm{d}A &= \int_0^{\pi/2}\int_0^1 abr^4(\cos^3\theta+\sin^3\theta)\mathrm{d}r\mathrm{d}\theta \\ &= ab\left(\int_0^1 r^4\mathrm{d}r\right)\left(\int_0^{\pi/2}(a^3\cos^3\theta+b^3\sin^3\theta)\mathrm{d}\theta\right) \\ &= ab\left(\frac{1}{5}\right)\left(\frac{2a^3+2b^3}{3}\right) \\ &= \frac{2ab(a^3+b^3)}{15}.\end{aligned}$$

◀

Homework

Problem 3.7.1 Evaluate $\int_{\mathcal{R}} xy\mathrm{d}A$ where $\mathcal{R}$ is the region

$$\mathcal{R}=\left\{(x,y)\in\mathbb{R}^2: x^2+y^2\leq 16, x\geq 1, y\geq 1\right\},$$

as in the figure 3.23. Set up the integral in both Cartesian and polar coordinates.

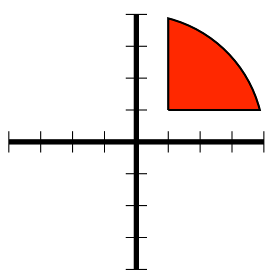

Figure 3.23: Problem 3.7.1.

Problem 3.7.2 Find

$$\int_D (x^2-y^2)\mathrm{d}A$$

where

$$D=\{(x,y)\in\mathbb{R}^2|(x-1)^2+y^2\leq 1\}.$$

Problem 3.7.3 Find

$$\int_D \sqrt{xy}\mathrm{d}A$$

where

$$D=\{(x,y)\in\mathbb{R}^2|(x^2+y^2)^2\leq 2xy\}.$$

Problem 3.7.4 Find $\int_D f\mathrm{d}A$ where

$$D=\{(x,y)\in\mathbb{R}^2: b^2x^2+a^2y^2=a^2b^2, (a,b)\in]0;+\infty[\text{ fixed}\}$$

and $f(x,y)=x^3+y^3$.

Problem 3.7.5 Let $a > 0$ and $b > 0$. Prove that

$$\int_R \sqrt{\frac{a^2b^2 - a^2y^2 - b^2x^2}{a^2b^2 + a^2y^2 + b^2x^2}}\mathrm{d}A = \frac{\pi ab(\pi - 2)}{8},$$

where R is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the first quadrant.

Problem 3.7.6 Prove that

$$\int_R \frac{y}{\sqrt{x^2 + y^2}}\mathrm{d}x\mathrm{d}y = \sqrt{2} - 1,$$

where

$$R = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, o < y < x^2\}.$$

Problem 3.7.7 Prove that the ellipse

$$(x - 2y + 3)^2 + (3x + 4y - 1)^2 = 4$$

bounds an area of $\frac{2\pi}{5}$.

Problem 3.7.8 Find

$$\int_D \sqrt{x^2 + y^2}\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x^2 + y^2 \leq 1, x^2 + y^2 - 2y \geq 0\}.$$

Problem 3.7.9 Find $\int_D f\mathrm{d}A$ where

$$D = \{(x, y) \in \mathbb{R}^2 | y \geq 0, x^2 + y^2 - 2x \leq 0\}$$

and $f(x, y) = x^2y$.

Problem 3.7.10 Let $D = \{(x, y) \in \mathbb{R}^2 : x \geq 1, x^2 + y^2 \leq 4\}$. Find $\int_D x\mathrm{d}A$.

Problem 3.7.11 Find $\int_D f\mathrm{d}A$ where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 1, x^2 + y^2 - 2x \leq 0\}$$

and $f(x, y) = \frac{1}{(x^2 + y^2)^2}$.

Problem 3.7.12 Let

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - y \leq 0, x^2 + y^2 - x \leq 0\}.$$

Find the integral

$$\int_D (x + y)^2\mathrm{d}A.$$

Problem 3.7.13 Let $D = \{(x, y) \in \mathbb{R}^2 | y \leq x^2 + y^2 \leq 1\}$. Compute

$$\int_D \frac{\mathrm{d}A}{(1 + x^2 + y^2)^2}.$$

Problem 3.7.14 Evaluate

$$\int_{\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^4 + y^4 \leq 1\}} x^3y^3\sqrt{1 - x^4 - y^4}\,\mathrm{d}A$$

using $x^2 = \rho\cos\theta$, $y^2 = \rho\sin\theta$.

Problem 3.7.15 William Thompson (Lord Kelvin) is credited to have said: "A mathematician is someone to whom

$$\int_0^{+\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

is as obvious as twice two is four to you. Liouville was a mathematician." Prove that

$$\int_0^{+\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

by following these steps.

❶ Let $a > 0$ be a real number and put $D_a = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq a^2\}$. Find

$$I_a = \int_{D_a} e^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y.$$

❷ Let $a > 0$ be a real number and put $\Delta_a = \{(x, y) \in \mathbb{R}^2 || x| \leq a, |y| \leq a\}$. Let

$$J_a = \int_{\Delta_a} e^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y.$$

Prove that

$$I_a \leq J_a \leq I_{a\sqrt{2}}.$$

❸ Deduce that

$$\int_0^{+\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Problem 3.7.16 Let $D = \{(x, y) \in \mathbb{R}^2 : 4 \leq x^2 + y^2 \leq 16\}$ and $f(x, y) = \dfrac{1}{x^2 + xy + y^2}$. Find $\displaystyle\int_D f(x, y)\mathrm{d}A$.

Problem 3.7.17 Prove that every closed convex region in the plane of area $\geq \pi$ has two points which are two units apart.

Problem 3.7.18 In the xy-plane, if R is the set of points inside and on a convex polygon, let $D(x, y)$ be the distance from (x, y) to the nearest point R. Show that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-D(x,y)}\,\mathrm{d}x\mathrm{d}y = 2\pi + L + A,$$

where L is the perimeter of R and A is the area of R.

3.8 Three-Manifolds

282 Definition A 3-*dimensional oriented manifold of* $\mathbb{R}^3$ is simply an open set (body) $V \in \mathbb{R}^3$, where the $+$ orientation is in the direction of the outward pointing normal to the body, and the $-$ orientation is in the direction of the inward pointing normal to the body. A general oriented 3-manifold is a union of open sets.

☞ *The region $-M$ has opposite orientation to M and*

$$\int_{-M} \omega = -\int_M \omega.$$

We will often write

$$\int_M f\mathrm{d}V$$

where $\mathrm{d}V$ *denotes the* volume element.

☞ *In this section, unless otherwise noticed, we will choose the positive orientation for the regions considered. This corresponds to using the volume form* $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Let $V \subseteq \mathbb{R}^3$. Given a function $f : V \to \mathbb{R}$, the integral

$$\int_V f \mathrm{d}V$$

is the sum of all the values of f restricted to V. In particular,

$$\int_V \mathrm{d}V$$

is the oriented volume of V.

283 Example Find

$$\int_{[0;1]^3} x^2 y e^{xyz} \, \mathrm{d}V.$$

Solution: ▶ *The integral is*

$$\begin{aligned} \int_0^1 \left(\int_0^1 \left(\int_0^1 x^2 y e^{xyz} \, \mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x &= \int_0^1 \left(\int_0^1 x(e^{xy} - 1) \, \mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 (e^x - x - 1) \mathrm{d}x \\ &= e - \frac{5}{2}. \end{aligned}$$

◀

284 Example Find $\int_R z \, \mathrm{d}V$ if

$$R = \{(x, y, z) \in \mathbb{R}^3 | x \geq 0, y \geq 0, z \geq 0, \sqrt{x} + \sqrt{y} + \sqrt{z} \leq 1\}.$$

Solution: ▶ *The integral is*

$$\begin{aligned} \int_R z \mathrm{d}x \mathrm{d}y \mathrm{d}z &= \int_0^1 z \left(\int_0^{(1-\sqrt{z})^2} \left(\int_0^{(1-\sqrt{z}-\sqrt{x})^2} \mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z \\ &= \int_0^1 z \left(\int_0^{(1-\sqrt{z})^2} (1 - \sqrt{z} - \sqrt{x})^2 \mathrm{d}x \right) \mathrm{d}z \\ &= \frac{1}{6} \int_0^1 z(1 - \sqrt{z})^4 \, \mathrm{d}z \\ &= \frac{1}{840}. \end{aligned}$$

◀

285 Example Prove that

$$\int_V x \mathrm{d}V = \frac{a^2 bc}{24},$$

where V is the tetrahedron

$$V = \left\{ (x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1 \right\}.$$

Solution: ► *We have*

$$\begin{aligned}\int_V x\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^c\int_0^{b-bz/c}\int_0^{a-ay/b-az/c} x\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \frac{1}{2}\int_0^c\int_0^{b-bz/c}\left(a-\frac{ay}{b}-\frac{az}{c}\right)^2\mathrm{d}y\mathrm{d}z\\ &= \frac{1}{6}\int_0^c\frac{a^2\left(-z+c\right)^3 b}{c^3}\mathrm{d}x\\ &= \frac{a^2bc}{24}\end{aligned}$$

◄

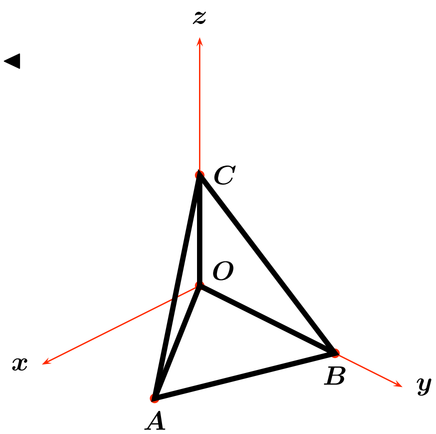

Figure 3.24: Problem 286.

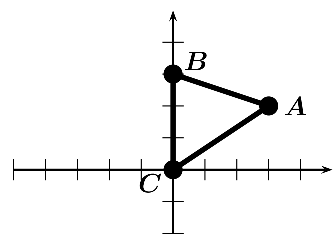

Figure 3.25: xy-projection.

286 Example Evaluate the integral $\int_S x\mathrm{d}V$ where S is the (unoriented) tetrahedron with vertices $(0,0,0)$, $(3,2,0)$, $(0,3,0)$, and $(0,0,2)$. See figure 3.24.

Solution: ► *A short computation shews that the plane passing through* $(3,2,0)$, $(0,3,0)$, *and* $(0,0,2)$ *has equation* $2x+6y+9z=18$. *Hence,* $0\leq z\leq\dfrac{18-2x-6y}{9}$. *We must now figure out the* xy *limits of integration. In figure 3.25 we draw the projection of the tetrahedron on the* xy *plane. The line passing through* AB *has equation* $y=-\dfrac{x}{3}+3$. *The line passing through* AC *has equation* $y=\dfrac{2}{3}x$.

We find, finally,

$$\begin{aligned}\int_S x\mathrm{d}V &= \int_0^3\int_{2x/3}^{3-x/3}\int_0^{(18-2x-6y)/9} x\mathrm{d}z\mathrm{d}y\mathrm{d}x\\ &= \int_0^3\int_{2x/3}^{3-x/3}\frac{18x-2x^2-6yx}{9}\mathrm{d}y\mathrm{d}x\\ &= \int_0^3\frac{18xy-2x^2y-3y^2x}{9}\Big|_{2x/3}^{3-x/3}\mathrm{d}x\\ &= \int_0^3\left(\frac{x^3}{3}-2x^2+3x\right)\mathrm{d}x\\ &= \frac{9}{4}\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> int(x,[x,y,z]=Tetrahedron(<0,0,0>,<3,2,0>,<0,3,0>,<0,0,2>));
```

◄

287 Example Evaluate $\int_R xyz\mathrm{d}V$, where R is the solid formed by the intersection of the parabolic cylinder $z = 4 - x^2$, the planes $z = 0$, $y = x$, and $y = 0$. Use the following orders of integration:

1. $\mathrm{d}z\mathrm{d}x\mathrm{d}y$
2. $\mathrm{d}x\mathrm{d}y\mathrm{d}z$

Solution: ▶ *We must find the projections of the solid on the the coordinate planes.*

1. *With the order $\mathrm{d}z\mathrm{d}x\mathrm{d}y$, the limits of integration of z can only depend, if at all, on x and y. Given an arbitrary point in the solid, its lowest z coordinate is 0 and its highest one is on the cylinder, so the limits for z are from $z = 0$ to $z = 4 - x^2$. The projection of the solid on the xy-plane is the area bounded by the lines $y = x$, $x = 2$, and the x and y axes.*

$$\begin{aligned}\int_0^2\int_0^y\int_0^{4-x^2} xyz\mathrm{d}z\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_0^2\int_0^y xy(4-x^2)^2\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\int_0^2\int_0^y y(16x - 8x^3 + x^5)\mathrm{d}x\mathrm{d}y \\ &= \int_0^2\left(4y^3 - y^5 + \frac{y^7}{12}\right)\mathrm{d}y \\ &= 8.\end{aligned}$$

2. *With the order $\mathrm{d}x\mathrm{d}y\mathrm{d}z$, the limits of integration of x can only depend, if at all, on y and z. Given an arbitrary point in the solid, x sweeps from the plane to $x = 2$, so the limits for x are from $x = y$ to $x = \sqrt{4-z}$. The projection of the solid on the yz-plane is the area bounded by $z = 4 - y^2$, and the z and y axes.*

$$\begin{aligned}\int_0^4\int_0^{\sqrt{4-z}}\int_y^2 xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \frac{1}{2}\int_0^4\int_0^{\sqrt{4-z}}(4y - y^3)z\mathrm{d}y\mathrm{d}z \\ &= \int_0^4\left(2z - \frac{z^3}{8}\right)\mathrm{d}z \\ &= 8.\end{aligned}$$

◀

Homework

Problem 3.8.1 Compute $\int_E z\mathrm{d}V$ where E is the region in the first octant bounded by the planes $y + z = 1$ and $x + z = 1$.

Problem 3.8.2 Consider the solid S in the first octant, bounded by the parabolic cylinder $z = 2 - \frac{x^2}{2}$ and the planes $z = 0$, $y = x$, and $y = 0$. Prove that $\int_S xyz = \frac{2}{3}$ first by integrating in the order $\mathrm{d}z\mathrm{d}y\mathrm{d}x$, and then by integrating in the order $\mathrm{d}y\mathrm{d}x\mathrm{d}z$.

Problem 3.8.3 Evaluate the integrals $\int_R 1\mathrm{d}V$ and $\int_R x\mathrm{d}V$, where R is the tetrahedron with vertices at $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 0)$, and $(0, 0, 1)$.

Problem 3.8.4 Compute $\int_E x\mathrm{d}V$ where E is the region in the first octant bounded by the plane $y = 3z$ and the cylinder $x^2 + y^2 = 9$.

Problem 3.8.5 Find $\int_D \frac{\mathrm{d}V}{(1+x^2z^2)(1+y^2z^2)}$ where

$$D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, z \geq 0\}.$$

3.9 Change of Variables

288 Example Find

$$\int_R (x+y+z)(x+y-z)(x-y-z)\mathrm{d}V,$$

where R is the tetrahedron bounded by the planes $x+y+z=0$, $x+y-z=0$, $x-y-z=0$, and $2x-z=1$.

Solution: ► *We make the change of variables*

$$u = x+y+z \implies \mathrm{d}u = \mathrm{d}x + \mathrm{d}y + \mathrm{d}z,$$

$$v = x+y-z \implies \mathrm{d}v = \mathrm{d}x + \mathrm{d}y - \mathrm{d}z,$$

$$w = x-y-z \implies \mathrm{d}w = \mathrm{d}x - \mathrm{d}y - \mathrm{d}z.$$

This gives

$$\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w = -4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

These forms have opposite orientations, so we choose, say,

$$\mathrm{d}u \wedge \mathrm{d}w \wedge \mathrm{d}v = 4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

which have the same orientation. Also,

$$2x - z = 1 \implies u + v + 2w = 2.$$

The tetrahedron in the xyz-coordinate frame is mapped into a tetrahedron bounded by $u = 0$, $v = 0$, $u + v + 2w = 1$ in the uvw-coordinate frame. The integral becomes

$$\frac{1}{4}\int_0^2 \int_0^{1-v/2} \int_0^{2-v-2w} uvw \, \mathrm{d}u\mathrm{d}w\mathrm{d}v = \frac{1}{180}.$$

Consider a transformation to cylindrical coordinates

$$(x, y, z) = (\rho\cos\theta, \rho\sin\theta, z).$$

From what we know about polar coordinates

$$\mathrm{d}x \wedge \mathrm{d}y = \rho\mathrm{d}\rho \wedge \mathrm{d}\theta.$$

Since the wedge product of forms is associative,

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \rho\mathrm{d}\rho \wedge \mathrm{d}\theta \wedge \mathrm{d}z.$$

◄

289 Example Find $\int_R z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$ if

$$R = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq 1, 0 \leq z \leq 1\}.$$

Solution: ► *The region of integration is mapped into*

$$\Delta = [0; 2\pi] \times [0; 1] \times [0; 1]$$

through a cylindrical coordinate change. The integral is therefore

$$\begin{aligned}\int_R f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \left(\int_0^{2\pi} \mathrm{d}\theta\right)\left(\int_0^1 \rho\,\mathrm{d}\rho\right)\left(\int_0^1 z^2\,\mathrm{d}z\right) \\ &= \frac{\pi}{3}.\end{aligned}$$

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290 Example Evaluate $\int_D (x^2 + y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ over the first octant region bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the planes $z = 0, z = 1, x = 0, x = y$.

Solution: ► *The integral is*

$$\int_0^1 \int_{\pi/4}^{\pi/2} \int_1^2 \rho^3 \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z = \frac{15\pi}{16}.$$

◄

291 Example Three long cylinders of radius R intersect at right angles. Find the volume of their intersection.

Solution: ► *Let* V *be the desired volume. By symmetry,* $V = 2^4 V'$, *where*

$$V' = \int_{D'} \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

$$D' = \{(x, y, z) \in \mathbb{R}^3 : 0 \le y \le x, 0 \le z, x^2 + y^2 \le R^2, y^2 + z^2 \le R^2, z^2 + x^2 \le R^2\}.$$

In this case it is easier to integrate with respect to z first. Using cylindrical coordinates

$$\Delta' = \left\{(\theta, \rho, z) \in \left[0; \frac{\pi}{4}\right] \times [0; R] \times [0; +\infty[, 0 \le z \le \sqrt{R^2 - \rho^2\cos^2\theta}\right\}.$$

Now,

$$\begin{aligned} V' &= \int_0^{\pi/4} \left(\int_0^R \left(\int_0^{\sqrt{R^2-\rho^2\cos^2\theta}} \mathrm{d}z\right)\rho\mathrm{d}\rho\right)\mathrm{d}\theta \\ &= \int_0^{\pi/4}\left(\int_0^R \rho\sqrt{R^2 - \rho^2\cos^2\theta}\mathrm{d}\rho\right)\mathrm{d}\theta \\ &= \int_0^{\pi/4} -\frac{1}{3\cos^2\theta}\left[(R^2 - \rho^2\cos^2\theta)^{3/2}\right]_0^R \mathrm{d}\theta \\ &= \frac{R^3}{3}\int_0^{\pi/4} \frac{1 - \sin^3\theta}{\cos^2\theta}\mathrm{d}\theta \\ \overset{=}{u = \cos\theta} &\quad \frac{R^3}{3}\left([\tan\theta]_0^{\pi/4} + \int_1^{\frac{\sqrt{2}}{2}} \frac{1 - u^2}{u^2}\mathrm{d}u\right) \\ &= \frac{R^3}{3}\left(1 - \left[u^{-1} + u\right]_1^{\frac{\sqrt{2}}{2}}\right) \\ &= \frac{\sqrt{2} - 1}{\sqrt{2}}R^3. \end{aligned}$$

Finally

$$V = 16V' = 8(2 - \sqrt{2})R^3.$$

◄

Consider now a change to spherical coordinates

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi.$$

We have

$$\begin{aligned}
\mathrm{d}x &= \cos\theta\sin\phi\mathrm{d}\rho - \rho\sin\theta\sin\phi\mathrm{d}\theta + \rho\cos\theta\cos\phi\mathrm{d}\phi,\\
\mathrm{d}y &= \sin\theta\sin\phi\mathrm{d}\rho + \rho\cos\theta\sin\phi\mathrm{d}\theta + \rho\sin\theta\cos\phi\mathrm{d}\phi,\\
\mathrm{d}z &= \cos\phi\mathrm{d}\rho - \rho\sin\phi\mathrm{d}\phi.
\end{aligned}$$

This gives

$$\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = -\rho^2\sin\phi\mathrm{d}\rho\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi.$$

From this derivation, the form $\mathrm{d}\rho\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi$ is negatively oriented, and so we choose

$$\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = \rho^2\sin\phi\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}\theta$$

instead.

292 Example Let $(a, b, c) \in]0; +\infty[^3$ be fixed. Find $\int_R xyz\ \mathrm{d}V$ if

$$R = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, x \geq 0, y \geq 0, z \geq 0\right\}.$$

Solution: ▶ *We use spherical coordinates, where*

$$(x, y, z) = (a\rho\cos\theta\sin\phi, b\rho\sin\theta\sin\phi, c\rho\cos\phi).$$

We have

$$\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = abc\rho^2\sin\phi\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}\rho.$$

The integration region is mapped into

$$\Delta = [0; 1] \times [0; \frac{\pi}{2}] \times [0; \frac{\pi}{2}].$$

The integral becomes

$$(abc)^2\left(\int_0^{\pi/2}\cos\theta\sin\theta\ \mathrm{d}\theta\right)\left(\int_0^1\rho^5\ \mathrm{d}\rho\right)\left(\int_0^{\pi/2}\cos^3\phi\sin\phi\ \mathrm{d}\phi\right) = \frac{(abc)^2}{48}.$$

◀

293 Example Let $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 9, 1 \leq z \leq 2\}$. Then

$$\begin{aligned}
\int_V \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{2\pi}\int_{\pi/2-\arcsin 2/3}^{\pi/2-\arcsin 1/3}\int_{1/\cos\phi}^{2/\cos\phi}\rho^2\sin\phi\ \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta\\
&= \frac{63\pi}{4}.
\end{aligned}$$

Homework

Problem 3.9.1 Consider the region $\mathcal{R}$ below the cone $z = \sqrt{x^2 + y^2}$ and above the paraboloid $z = x^2 + y^2$ for $0 \leq z \leq 1$. Set up integrals for the volume of this region in Cartesian, cylindrical and spherical coordinates. Also, find this volume.

Problem 3.9.2 Consider the integral $\int_{\mathcal{R}} x\mathrm{d}V$, where $\mathcal{R}$ is the region above the paraboloid $z = x^2 + y^2$ and under the sphere $x^2 + y^2 + z^2 = 4$. Set up integrals for the volume of this region in Cartesian, cylindrical and spherical coordinates. Also, find this volume.

Problem 3.9.3 Consider the region $\mathcal{R}$ bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$. Set up integrals for the volume of this region in Cartesian, cylindrical and spherical coordinates. Also, find this volume.

Problem 3.9.4 Prove that the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\frac{4\pi abc}{3}$. Here $a > 0$, $b > 0$, $c > 0$.

Problem 3.9.5 Compute $\int_E y\mathrm{d}V$ where E is the region between the cylinders $x^2+y^2 = 1$ and $x^2 + y^2 = 4$, below the plane $x - z = -2$ and above the xy-plane.

Problem 3.9.6 Prove that

$$\int_{\substack{x\geq 0, y\geq 0\\ x^2+y^2+z^2\leq R^2}} e^{-\sqrt{x^2+y^2+z^2}}\mathrm{d}V = \pi\left(2 - 2e^{-R} - 2Re^{-R} - R^2e^{-R}\right).$$

Problem 3.9.7 Compute $\int_E y^2z^2\mathrm{d}V$ where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

Problem 3.9.8 Compute $\int_E z\sqrt{x^2 + y^2 + z^2}\mathrm{d}V$ where E is is the upper solid hemisphere bounded by the xy-plane and the sphere of radius 1 about the origin.

Problem 3.9.9 Compute the 4-dimensional integral

$$\iiiint_{x^2+y^2+u^2+v^2\leq 1} e^{x^2+y^2+u^2+v^2}\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v.$$

Problem 3.9.10 (Putnam Exam 1984) Find

$$\int_R x^1y^9z^8(1 - x - y - z)^4\ \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$R = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}.$$

3.10 Surface Integrals

294 Definition A 2-*dimensional oriented manifold of* $\mathbb{R}^3$ is simply a smooth surface $D \in \mathbb{R}^3$, where the $+$ orientation is in the direction of the outward normal pointing away from the origin and the $-$ orientation is in the direction of the inward normal pointing towards the origin. A general oriented 2-manifold in $\mathbb{R}^3$ is a union of surfaces.

☞ *The surface $-\Sigma$ has opposite orientation to Σ and*

$$\int_{-\Sigma}\omega = -\int_{\Sigma}\omega.$$

☞ *In this section, unless otherwise noticed, we will choose the positive orientation for the regions considered. This corresponds to using the ordered basis*

$$\{\mathrm{d}y\wedge\mathrm{d}z,\ \mathrm{d}z\wedge\mathrm{d}x,\ \mathrm{d}x\wedge\mathrm{d}y\}.$$

295 Definition Let $f : \mathbb{R}^3 \to \mathbb{R}$. The integral of f over the smooth surface Σ (oriented in the positive sense) is given by the expression

$$\int_{\Sigma} f \left|\left|\mathrm{d}^2\mathbf{x}\right|\right|.$$

Here

$$\left|\left|\mathrm{d}^2\mathbf{x}\right|\right| = \sqrt{(\mathrm{d}x \wedge \mathrm{d}y)^2 + (\mathrm{d}z \wedge \mathrm{d}x)^2 + (\mathrm{d}y \wedge \mathrm{d}z)^2}$$

is the *surface area element.*

296 Example Evaluate $\int_{\Sigma} z\left|\left|\mathrm{d}^2\mathbf{x}\right|\right|$ where Σ is the outer surface of the section of the paraboloid $z = x^2 + y^2, 0 \le z \le 1$.

Solution: ► *We parametrise the paraboloid as follows. Let* $x = u, y = v, z = u^2 + v^2$. *Observe that the domain* D *of* Σ *is the unit disk* $u^2 + v^2 \le 1$. *We see that*

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}u \wedge \mathrm{d}v,$$

$$\mathrm{d}y \wedge \mathrm{d}z = -2u\mathrm{d}u \wedge \mathrm{d}v,$$

$$\mathrm{d}z \wedge \mathrm{d}x = -2v\mathrm{d}u \wedge \mathrm{d}v,$$

and so

$$\left|\left|\mathrm{d}^2\mathbf{x}\right|\right| = \sqrt{1 + 4u^2 + 4v^2}\mathrm{d}u \wedge \mathrm{d}v.$$

Now,

$$\int_{\Sigma} z\left|\left|\mathrm{d}^2\mathbf{x}\right|\right| = \int_{D} (u^2 + v^2)\sqrt{1 + 4u^2 + 4v^2}\mathrm{d}u\mathrm{d}v.$$

To evaluate this last integral we use polar coordinates, and so

$$\begin{aligned}\int_{D} (u^2 + v^2)\sqrt{1 + 4u^2 + 4v^2}\mathrm{d}u\mathrm{d}v &= \int_0^{2\pi}\int_0^1 \rho^3\sqrt{1 + 4\rho^2}\mathrm{d}\rho\mathrm{d}\theta \\ &= \frac{\pi}{12}(5\sqrt{5} + \frac{1}{5}).\end{aligned}$$

◄

297 Example Find the area of that part of the cylinder $x^2+y^2 = 2y$ lying inside the sphere $x^2+y^2+z^2 = 4$.

Solution: ► *We have*

$$x^2 + y^2 = 2y \iff x^2 + (y-1)^2 = 1.$$

We parametrise the cylinder by putting $x = \cos u, y - 1 = \sin u$, *and* $z = v$. *Hence*

$$\mathrm{d}x = -\sin u\mathrm{d}u,\ \mathrm{d}y = \cos u\mathrm{d}u,\ \mathrm{d}z = dv,$$

whence

$$\mathrm{d}x \wedge \mathrm{d}y = 0, \mathrm{d}y \wedge \mathrm{d}z = \cos u\mathrm{d}u \wedge \mathrm{d}v, \mathrm{d}z \wedge \mathrm{d}x = \sin u\mathrm{d}u \wedge \mathrm{d}v,$$

and so

$$\begin{aligned}\left|\left|\mathrm{d}^2\mathbf{x}\right|\right| &= \sqrt{(\mathrm{d}x \wedge \mathrm{d}y)^2 + (\mathrm{d}z \wedge \mathrm{d}x)^2 + (\mathrm{d}y \wedge \mathrm{d}z)^2} \\ &= \sqrt{\cos^2 u + \sin^2 u}\,\mathrm{d}u \wedge \mathrm{d}v \\ &= \mathrm{d}u \wedge \mathrm{d}v.\end{aligned}$$

The cylinder and the sphere intersect when $x^2+y^2=2y$ and $x^2+y^2+z^2=4$, that is, when $z^2=4-2y$, i.e. $v^2=4-2(1+\sin u)=2-2\sin u$. Also $0\leq u\leq\pi$. The integral is thus

$$\begin{aligned}\int_{\Sigma}\left|\left|\mathrm{d}^2\mathrm{x}\right|\right| &= \int_0^{\pi}\int_{-\sqrt{2-2\sin u}}^{\sqrt{2-2\sin u}}\mathrm{d}v\mathrm{d}u=\int_0^{\pi}2\sqrt{2-2\sin u}\mathrm{d}u\\ &= 2\sqrt{2}\int_0^{\pi}\sqrt{1-\sin u}\,\mathrm{d}u\\ &= 2\sqrt{2}\left(4\sqrt{2}-4\right).\end{aligned}$$

◀

298 Example Evaluate

$$\int_{\Sigma}x\mathrm{d}y\mathrm{d}z+(z^2-zx)\mathrm{d}z\mathrm{d}x-xy\mathrm{d}x\mathrm{d}y,$$

where Σ is the top side of the triangle with vertices at $(2,0,0)$, $(0,2,0)$, $(0,0,4)$.

Solution: ► *Observe that the plane passing through the three given points has equation $2x+2y+z=4$. We project this plane onto the coordinate axes obtaining*

$$\int_{\Sigma}x\mathrm{d}y\mathrm{d}z=\int_0^4\int_0^{2-z/2}(2-y-z/2)\mathrm{d}y\mathrm{d}z=\frac{8}{3},$$

$$\int_{\Sigma}(z^2-zx)\mathrm{d}z\mathrm{d}x=\int_0^2\int_0^{4-2x}(z^2-zx)\mathrm{d}z\mathrm{d}x=8,$$

$$-\int_{\Sigma}xy\mathrm{d}x\mathrm{d}y=-\int_0^2\int_0^{2-y}xy\mathrm{d}x\mathrm{d}y=-\frac{2}{3},$$

and hence

$$\int_{\Sigma}x\mathrm{d}y\mathrm{d}z+(z^2-zx)\mathrm{d}z\mathrm{d}x-xy\mathrm{d}x\mathrm{d}y=10.$$

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Homework

Problem 3.10.1 Evaluate $\int_{\Sigma}y\left|\left|\mathrm{d}^2\mathrm{x}\right|\right|$ where Σ is the surface $z=x+y^2, 0\leq x\leq 1, 0\leq y\leq 2$.

Problem 3.10.2 Consider the cone $z=\sqrt{x^2+y^2}$. Find the surface area of the part of the cone which lies between the planes $z=1$ and $z=2$.

Problem 3.10.3 Evaluate $\int_{\Sigma}x^2\left|\left|\mathrm{d}^2\mathrm{x}\right|\right|$ where Σ is the surface of the unit sphere $x^2+y^2+z^2=1$.

Problem 3.10.4 Evaluate $\int_S z\left|\left|\mathrm{d}^2\mathrm{x}\right|\right|$ over the conical surface $z=\sqrt{x^2+y^2}$ between $z=0$ and $z=1$.

Problem 3.10.5 You put a perfectly spherical egg through an egg slicer, resulting in n slices of identical height, but you forgot to peel it first! Shew that the amount of egg shell in any of the slices is the same. Your argument must use surface integrals.

Problem 3.10.6 Evaluate
$$\int_{\Sigma} xy\mathrm{d}y\mathrm{d}z - x^2\mathrm{d}z\mathrm{d}x + (x+z)\mathrm{d}x\mathrm{d}y,$$
where Σ is the top of the triangular region of the plane $2x+2y+z=6$ bounded by the first octant.

3.11 Green's, Stokes', and Gauss' Theorems

We are now in position to state the general Stoke's Theorem.

299 Theorem (General Stoke's Theorem) Let M be a smooth oriented manifold, having boundary ∂M. If ω is a differential form, then
$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega.$$

In $\mathbb{R}^2$, if ω is a 1-form, this takes the name of *Green's Theorem.*

300 Example Evaluate $\oint_C (x-y^3)\mathrm{d}x + x^3\mathrm{d}y$ where C is the circle $x^2+y^2=1$.

Solution: ▸ *We will first use Green's Theorem and then evaluate the integral directly. We have*
$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}(x-y^3)\wedge\mathrm{d}x + \mathrm{d}(x^3)\wedge\mathrm{d}y\\ &= (\mathrm{d}x - 3y^2\mathrm{d}y)\wedge\mathrm{d}x + (3x^2\mathrm{d}x)\wedge\mathrm{d}y\\ &= (3y^2+3x^2)\mathrm{d}x\wedge\mathrm{d}y.\end{aligned}$$

The region M is the area enclosed by the circle $x^2+y^2=1$. Thus by Green's Theorem, and using polar coordinates,
$$\begin{aligned}\oint_C (x-y^3)\mathrm{d}x + x^3\mathrm{d}y &= \int_M (3y^2+3x^2)\mathrm{d}x\mathrm{d}y\\ &= \int_0^{2\pi}\int_0^1 3\rho^2\rho\mathrm{d}\rho\mathrm{d}\theta\\ &= \frac{3\pi}{2}.\end{aligned}$$

Aliter: *We can evaluate this integral directly, again resorting to polar coordinates.*
$$\begin{aligned}\oint_C (x-y^3)\mathrm{d}x + x^3\mathrm{d}y &= \int_0^{2\pi} (\cos\theta - \sin^3\theta)(-\sin\theta)\mathrm{d}\theta + (\cos^3\theta)(\cos\theta)\mathrm{d}\theta\\ &= \int_0^{2\pi} (\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta)\mathrm{d}\theta.\end{aligned}$$

To evaluate the last integral, observe that $1 = (\sin^2\theta+\cos^2\theta)^2 = \sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta$, whence the integral equals
$$\begin{aligned}\int_0^{2\pi} (\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta)\mathrm{d}\theta &= \int_0^{2\pi} (1 - 2\sin^2\theta\cos^2\theta - \sin\theta\cos\theta)\mathrm{d}\theta\\ &= \frac{3\pi}{2}.\end{aligned}$$

◂

In general, let
$$\omega = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$$

be a 1-form in $\mathbb{R}^2$. Then

$$\begin{aligned} d\omega &= df(x,y)\wedge dx + dg(x,y)\wedge dy \\ &= \left(\frac{\partial}{\partial x}f(x,y)dx + \frac{\partial}{\partial y}f(x,y)dy\right)\wedge dx + \left(\frac{\partial}{\partial x}g(x,y)dx + \frac{\partial}{\partial y}g(x,y)dy\right)\wedge dy \\ &= \left(\frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}f(x,y)\right)dx\wedge dy \end{aligned}$$

which gives the classical Green's Theorem

$$\int_{\partial M} f(x,y)dx + g(x,y)dy = \int_M \left(\frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}f(x,y)\right)dxdy.$$

In $\mathbb{R}^3$, if ω is a 2-form, the above theorem takes the name of *Gauß'* or the *Divergence Theorem.*

301 Example Evaluate $\int_S (x-y)dydz + zdzdx - ydxdy$ where S is the surface of the sphere

$$x^2+y^2+z^2=9$$

and the positive direction is the outward normal.

Solution: ► *The region M is the interior of the sphere $x^2+y^2+z^2=9$. Now,*

$$\begin{aligned} d\omega &= (dx-dy)\wedge dy\wedge dz + dz\wedge dz\wedge dx - dy\wedge dx\wedge dy \\ &= dx\wedge dy\wedge dz. \end{aligned}$$

The integral becomes

$$\begin{aligned} \int_M dxdydz &= \frac{4\pi}{3}(27) \\ &= 36\pi. \end{aligned}$$

Aliter: *We could evaluate this integral directly. We have*

$$\int_\Sigma (x-y)dydz = \int_\Sigma xdydz,$$

since $(x,y,z)\mapsto -y$ is an odd function of y and the domain of integration is symmetric with respect to y. Now,

$$\begin{aligned} \int_\Sigma xdydz &= \int_{-3}^{3}\int_0^{2\pi} |\rho|\sqrt{9-\rho^2}d\rho d\theta \\ &= 36\pi. \end{aligned}$$

Also

$$\int_\Sigma zdzdx = 0,$$

since $(x,y,z)\mapsto z$ is an odd function of z and the domain of integration is symmetric with respect to z. Similarly

$$\int_\Sigma -ydxdy = 0,$$

since $(x,y,z)\mapsto -y$ is an odd function of y and the domain of integration is symmetric with respect to y. ◄

In general, let

$$\omega = f(x,y,z)dy\wedge dz + g(x,y,z)dz\wedge dx + h(x,y,z)dx\wedge dy$$

be a 2-form in $\mathbb{R}^3$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f(x,y,z)\mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}g(x,y,z)\mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}h(x,y,z)\mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial}{\partial x}f(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}f(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}f(x,y,z)\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\quad + \left(\frac{\partial}{\partial x}g(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}g(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}g(x,y,z)\mathrm{d}z\right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial}{\partial x}h(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}h(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}h(x,y,z)\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial}{\partial x}f(x,y,z) + \frac{\partial}{\partial y}g(x,y,z) + \frac{\partial}{\partial z}h(x,y,z)\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,
\end{aligned}$$

which gives the classical Gauss's Theorem

$$\int_{\partial M} f(x,y,z)\mathrm{d}y\mathrm{d}z + g(x,y,z)\mathrm{d}z\mathrm{d}x + h(x,y,z)\mathrm{d}x\mathrm{d}y = \int_{M} \left(\frac{\partial}{\partial x}f(x,y,z) + \frac{\partial}{\partial y}g(x,y,z) + \frac{\partial}{\partial z}h(x,y,z)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Using classical notation, if

$$\overrightarrow{\mathrm{a}} = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix}, \mathrm{d}\overrightarrow{\mathrm{S}} = \begin{bmatrix} \mathrm{d}y\mathrm{d}z \\ \mathrm{d}z\mathrm{d}x \\ \mathrm{d}x\mathrm{d}y \end{bmatrix},$$

then

$$\int_{M} (\nabla \bullet \overrightarrow{\mathrm{a}})\mathrm{d}V = \int_{\partial M} \overrightarrow{\mathrm{a}} \bullet \mathrm{d}\overrightarrow{\mathrm{S}}.$$

The classical Stokes' Theorem occurs when ω is a 1-form in $\mathbb{R}^3$.

302 Example Evaluate $\oint_C y\mathrm{d}x + (2x - z)\mathrm{d}y + (z - x)\mathrm{d}z$ where C is the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$.

Solution: ► *We have*

$$\begin{aligned}
\mathrm{d}\omega &= (\mathrm{d}y) \wedge \mathrm{d}x + (2\mathrm{d}x - \mathrm{d}z) \wedge \mathrm{d}y + (\mathrm{d}z - \mathrm{d}x) \wedge \mathrm{d}z \\
&= -\mathrm{d}x \wedge \mathrm{d}y + 2\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}x \\
&= \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}x.
\end{aligned}$$

Since on C, $z = 1$, the surface Σ on which we are integrating is the inside of the circle $x^2 + y^2 + 1 = 4$, i.e., $x^2 + y^2 = 3$. Also, $z = 1$ implies $\mathrm{d}z = 0$ and so

$$\int_{\Sigma} \mathrm{d}\omega = \int_{\Sigma} \mathrm{d}x\mathrm{d}y.$$

Since this is just the area of the circular region $x^2 + y^2 \leq 3$, the integral evaluates to

$$\int_{\Sigma} \mathrm{d}x\mathrm{d}y = 3\pi.$$

◄

In general, let

$$\omega = f(x,y,z)\mathrm{d}x + g(x,y,z)\mathrm{d}y + +h(x,y,z)\mathrm{d}z$$

be a 1-form in $\mathbb{R}^3$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f(x,y,z)\wedge \mathrm{d}x + \mathrm{d}g(x,y,z)\wedge \mathrm{d}y + \mathrm{d}h(x,y,z)\wedge \mathrm{d}z \\
&= \left(\frac{\partial}{\partial x}f(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}f(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}f(x,y,z)\mathrm{d}z\right)\wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial}{\partial x}g(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}g(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}g(x,y,z)\mathrm{d}z\right)\wedge \mathrm{d}y \\
&\quad\quad + \left(\frac{\partial}{\partial x}h(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}h(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}h(x,y,z)\mathrm{d}z\right)\wedge \mathrm{d}z \\
&= \left(\frac{\partial}{\partial y}h(x,y,z) - \frac{\partial}{\partial z}g(x,y,z)\right)\mathrm{d}y\wedge \mathrm{d}z \\
&\quad\quad + \left(\frac{\partial}{\partial z}f(x,y,z) - \frac{\partial}{\partial x}h(x,y,z)\right)\mathrm{d}z\wedge \mathrm{d}x \\
&\quad\quad \left(\frac{\partial}{\partial x}g(x,y,z) - \frac{\partial}{\partial y}f(x,y,z)\right)\mathrm{d}x\wedge \mathrm{d}y
\end{aligned}$$

which gives the classical Stokes' Theorem

$$\begin{aligned}
\int_{\partial M} f(x,y,z)\mathrm{d}x + g(x,y,z)\mathrm{d}y + h(x,y,z)\mathrm{d}z \\
= \int_M \left(\frac{\partial}{\partial y}h(x,y,z) - \frac{\partial}{\partial z}g(x,y,z)\right)\mathrm{d}y\mathrm{d}z \\
+ \left(\frac{\partial}{\partial z}g(x,y,z) - \frac{\partial}{\partial x}f(x,y,z)\right)\mathrm{d}x\mathrm{d}y \\
+ \left(\frac{\partial}{\partial x}h(x,y,z) - \frac{\partial}{\partial y}f(x,y,z)\right)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Using classical notation, if

$$\overrightarrow{\mathrm{a}} = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix}, \quad \mathrm{d}\overrightarrow{\mathrm{r}} = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix}, \quad \mathrm{d}\overrightarrow{\mathrm{S}} = \begin{bmatrix} \mathrm{d}y\mathrm{d}z \\ \mathrm{d}z\mathrm{d}x \\ \mathrm{d}x\mathrm{d}y \end{bmatrix},$$

then

$$\int_M (\nabla \times \overrightarrow{\mathrm{a}}) \bullet \mathrm{d}\overrightarrow{\mathrm{S}} = \int_{\partial M} \overrightarrow{\mathrm{a}} \bullet \mathrm{d}\overrightarrow{\mathrm{r}}.$$

Homework

Problem 3.11.1 Evaluate $\oint_C x^3y\mathrm{d}x + xy\mathrm{d}y$ where C is the square with vertices at $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

Problem 3.11.2 Consider the triangle $\triangle$ with vertices $A : (0,0)$, $B : (1,1)$, $C : (-2,2)$.

❶ If L_{PQ} denotes the equation of the line joining P and Q find L_{AB}, L_{AC}, and L_{BC}.

❷ Evaluate

$$\oint_{\triangle} y^2\mathrm{d}x + x\mathrm{d}y.$$

❸ Find

$$\int_{\mathscr{D}} (1-2y)\mathrm{d}x\wedge \mathrm{d}y$$

where $\mathscr{D}$ is the interior of $\triangle$.

Problem 3.11.3 Problems 1 through 4 refer to the differential form

$$\omega = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + 2z\mathrm{d}x \wedge \mathrm{d}y,$$

and the solid M whose boundaries are the paraboloid $z = 1 - x^2 - y^2$, $0 \leq z \leq 1$ and the disc $x^2 + y^2 \leq 1$, $z = 0$. The surface ∂M of the solid is positively oriented upon considering outward normals.

1. Prove that $\mathrm{d}\omega = 4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.
2. Prove that in Cartesian coordinates, $\displaystyle\int_{\partial M} \omega = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} 4\mathrm{d}z\mathrm{d}y\mathrm{d}x.$
3. Prove that in cylindrical coordinates, $\displaystyle\int_{M} \mathrm{d}\omega = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{1-r^2} 4r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta.$
4. Prove that $\displaystyle\int_{\partial M} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + 2z\mathrm{d}x\mathrm{d}y = 2\pi.$

Problem 3.11.4 Problems 1 through 4 refer to the box

$$M = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 2\},$$

the upper face of the box

$$U = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ z = 2\},$$

the boundary of the box without the upper top $S = \partial M \setminus U$, and the differential form

$$\omega = (\arctan y - x^2)\mathrm{d}y \wedge \mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z \wedge \mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x \wedge \mathrm{d}y.$$

1. Prove that $\mathrm{d}\omega = 3y^2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.
2. Prove that $\displaystyle\int_{\partial M} (\arctan y - x^2)\mathrm{d}y\mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z\mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x\mathrm{d}y = \int_{0}^{2} \int_{0}^{1} \int_{0}^{1} 3y^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = 2$. Here the boundary of the box is positively oriented considering outward normals.
3. Prove that the integral on the upper face of the box is $\displaystyle\int_{U} (\arctan y - x^2)\mathrm{d}y\mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z\mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x\mathrm{d}y = \int_{0}^{1} \int_{0}^{1} 4x + 12y^2\mathrm{d}x\mathrm{d}y = 6.$
4. Prove that the integral on the open box is $\displaystyle\int_{\partial M \setminus U} (\arctan y - x^2)\mathrm{d}y\mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z\mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x\mathrm{d}y = -4.$

Problem 3.11.5 Problems 1 through 3 refer to a triangular surface T in $\mathbb{R}^3$ and a differential form ω. The vertices of T are at $A(6, 0, 0)$, $B(0, 12, 0)$, and $C(0, 0, 3)$. The boundary of of the triangle ∂T is oriented positively by starting at A, continuing to B, following to C, and ending again at A. The surface T is oriented positively by considering the top of the triangle, as viewed from a point far above the triangle. The differential form is

$$\omega = (2xz + \arctan e^x)\,\mathrm{d}x + (xz + (y+1)^y)\,\mathrm{d}y + \left(xy + \frac{y^2}{2} + \log(1 + z^2)\right)\mathrm{d}z.$$

1. Prove that the equation of the plane that contains the triangle T is $2x + y + 4z = 12$.
2. Prove that $\mathrm{d}\omega = y\mathrm{d}y \wedge \mathrm{d}z + (2x - y)\,\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$.
3. Prove that $\displaystyle\int_{\partial T} (2xz + \arctan e^x)\,\mathrm{d}x + (xz + (y+1)^y)\,\mathrm{d}y + \left(xy + \frac{y^2}{2} + \log(1 + z^2)\right)\mathrm{d}z = \int_{0}^{3} \int_{0}^{12-4z} y\mathrm{d}y\mathrm{d}z + \int_{0}^{6} \int_{0}^{3-x/2} 2x\mathrm{d}z\mathrm{d}x = 108.$

Problem 3.11.6 Use Green's Theorem to prove that

$$\int_{\Gamma} (x^2 + 2y^3)\mathrm{d}y = 16\pi,$$

where Γ is the circle $(x - 2)^2 + y^2 = 4$. Also, prove this directly by using a path integral.

Problem 3.11.7 Let Γ denote the curve of intersection of the plane $x+y=2$ and the sphere $x^2-2x+y^2-2y+z^2=0$, oriented clockwise when viewed from the origin. Use Stoke's Theorem to prove that

$$\int_{\Gamma} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = -2\pi\sqrt{2}.$$

Prove this directly by parametrising the boundary of the surface and evaluating the path integral.

Problem 3.11.8 Use Green's Theorem to evaluate

$$\oint_C (x^3-y^3)\mathrm{d}x + (x^3+y^3)\mathrm{d}y,$$

where C is the positively oriented boundary of the region between the circles $x^2+y^2=2$ and $x^2+y^2=4$.

A Answers and Hints

1.1.1 No. The zero vector $\overrightarrow{0}$, has magnitude but no direction.

1.1.2 We have $2\overrightarrow{BC} = \overrightarrow{BE} + \overrightarrow{EC}$. By Chasles' Rule $\overrightarrow{AC} = \overrightarrow{AE} + \overrightarrow{EC}$, and $\overrightarrow{BD} = \overrightarrow{BE} + \overrightarrow{ED}$. We deduce that

$$\overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AE} + \overrightarrow{EC} + \overrightarrow{BE} + \overrightarrow{ED} = \overrightarrow{AD} + \overrightarrow{BC}.$$

But since $ABCD$ is a parallelogram, $\overrightarrow{AD} = \overrightarrow{BC}$. Hence

$$\overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{BC} = 2\overrightarrow{BC}.$$

1.1.4 We have $\overrightarrow{IA} = -3\overrightarrow{IB} \iff \overrightarrow{IA} = -3(\overrightarrow{IA} + \overrightarrow{AB}) = -3\overrightarrow{IA} - 3\overrightarrow{AB}$. Thus we deduce

$$\begin{aligned}
\overrightarrow{IA} + 3\overrightarrow{IA} = -3\overrightarrow{AB} &\iff 4\overrightarrow{IA} = -3\overrightarrow{AB} \\
&\iff 4\overrightarrow{AI} = 3\overrightarrow{AB} \\
&\iff \overrightarrow{AI} = \frac{3}{4}\overrightarrow{AB}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\overrightarrow{JA} = -\frac{1}{3}\overrightarrow{JB} &\iff 3\overrightarrow{JA} = -\overrightarrow{JB} \\
&\iff 3\overrightarrow{JA} = -\overrightarrow{JA} - \overrightarrow{AB} \\
&\iff 4\overrightarrow{JA} = -\overrightarrow{AB} \\
&\iff \overrightarrow{AJ} = \frac{1}{4}\overrightarrow{AB}
\end{aligned}$$

.

Thus we take $\mathbf{I}$ such that $\overrightarrow{AI} = \frac{3}{4}\overrightarrow{AB}$ and $\mathbf{J}$ such that $\overrightarrow{AJ} = \frac{1}{4}\overrightarrow{AB}$.

Now

$$\begin{aligned}
\overrightarrow{MA} + 3\overrightarrow{MB} &= \overrightarrow{MI} + \overrightarrow{IA} + 3\overrightarrow{IB} \\
&= 4\overrightarrow{MI} + \overrightarrow{IA} + 3\overrightarrow{IB} \\
&= 4\overrightarrow{MI},
\end{aligned}$$

and

$$\begin{aligned}
3\overrightarrow{MA} + \overrightarrow{MB} &= 3\overrightarrow{MJ} + 3\overrightarrow{JA} + \overrightarrow{MJ} + \overrightarrow{JB} \\
&= 4\overrightarrow{MJ} + 3\overrightarrow{JA} + \overrightarrow{JB} \\
&= 4\overrightarrow{MJ}.
\end{aligned}$$

1.1.5

$$x + 1 = t = 2 - y \implies y = -x + 1.$$

1.1.6 $\alpha = \frac{4}{7}, \beta = \frac{3}{7}, l = \frac{4}{9}, m = \frac{2}{9}, n = \frac{1}{3}$.

1.1.8 [A]. $\overrightarrow{0}$, [B]. $\overrightarrow{0}$, [C]. $\overrightarrow{0}$, [D]. $\overrightarrow{0}$, [E]. $2\overrightarrow{c}(= 2\overrightarrow{d})$

1.3.2 Plainly,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{b-a}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{a+b}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

1.4.1 $a = -3, b = -\frac{1}{2}$.

1.4.2 The desired transformations are in figures A.1 through A.3.

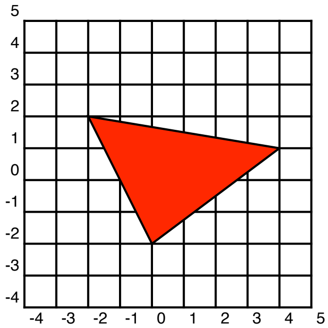

Figure A.1: Horizontal Stretch.

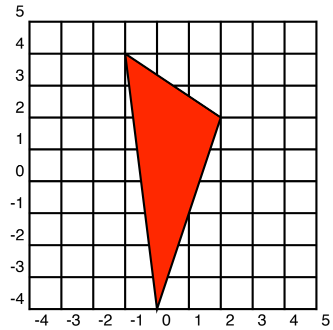

Figure A.2: Vertical Stretch.

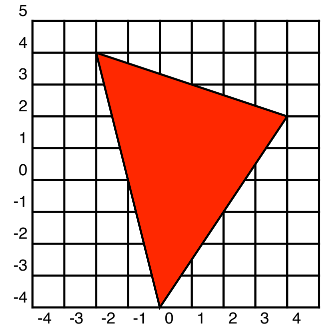

Figure A.3: Horizontal and Vertical Stretch.

1.4.3 The desired transformations are shewn in figures through A.4 A.7.

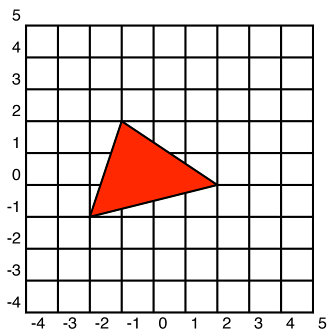

Figure A.4: Levogyrate rotation $\frac{\pi}{2}$ radians.

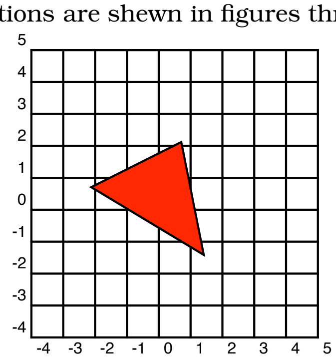

Figure A.5: Levogyrate rotation $\frac{\pi}{4}$ radians.

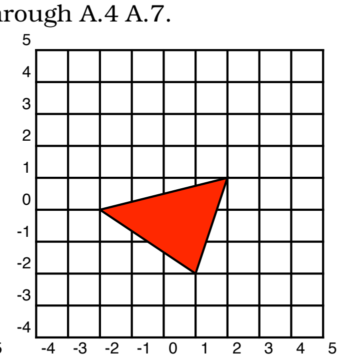

Figure A.6: Dextrogyrate rotation $\frac{\pi}{2}$ radians.

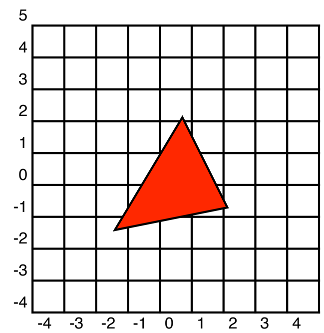

Figure A.7: Dextrogyrate rotation $\frac{\pi}{4}$ radians.

1.4.8 The transformations are shewn in figures A.8 through A.10.

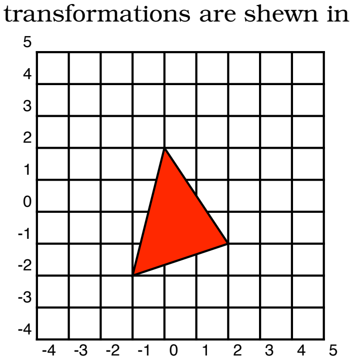

Figure A.8: Reflexion about the x-axis.

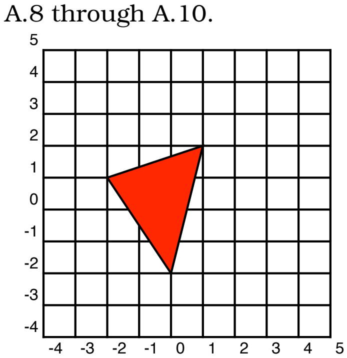

Figure A.9: Reflexion about the y-axis .

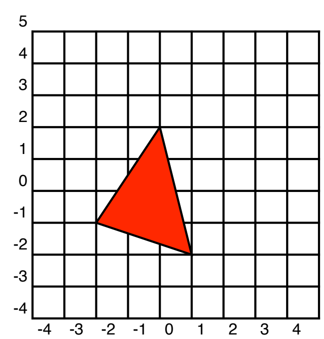

Figure A.10: Reflexion about the origin.

1.4.9 $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, $bc = -a^2$

1.5.1 Upon solving the equations

$$-3a_1 + 2a_2 = 0, \qquad a_1^2 + a_2^2 = 13$$

we find $(a_1, a_2) = (2, 3)$ or $(a_1, a_2) = (-2, -3)$.

1.5.2 Since $\overrightarrow{a} \bullet \overrightarrow{b} = 0$, we have

$$\begin{aligned} ||\overrightarrow{a} + \overrightarrow{b}||^2 &= (\overrightarrow{a} + \overrightarrow{b})\bullet(\overrightarrow{a} + \overrightarrow{b}) \\ &= \overrightarrow{a}\bullet\overrightarrow{a} + 2\overrightarrow{a}\bullet\overrightarrow{b} + \overrightarrow{b}\bullet\overrightarrow{b} \\ &= \overrightarrow{a}\bullet\overrightarrow{a} + 0 + \overrightarrow{b}\bullet\overrightarrow{b} \\ &= ||\overrightarrow{a}||^2 + ||\overrightarrow{b}||^2, \end{aligned}$$

from where the desired result follows.

1.5.3 By the CBS Inequality,

$$(a^2 \cdot 1 + b^2 \cdot 1) \leq (a^4 + b^4)^{1/2}(1^2 + 1^2)^{1/2},$$

whence the assertion follows.

1.5.4 We have $\forall \overrightarrow{v} \in \mathbb{R}^2, \overrightarrow{v} \bullet (\overrightarrow{a} - \overrightarrow{b}) = 0.$ In particular, choosing $\overrightarrow{v} = \overrightarrow{a} - \overrightarrow{b}$, we gather

$$(\overrightarrow{a} - \overrightarrow{b})\bullet(\overrightarrow{a} - \overrightarrow{b}) = ||\overrightarrow{a} - \overrightarrow{b}||^2 = 0.$$

But the norm of a vector is 0 if and only if the vector is the $\overrightarrow{0}$ vector. Therefore $\overrightarrow{a} - \overrightarrow{b} = \overrightarrow{0}$, i.e., $\overrightarrow{a} = \overrightarrow{b}$.

1.5.5 We have

$$\begin{aligned} ||\overrightarrow{u} + \overrightarrow{v}||^2 - ||\overrightarrow{u} - \overrightarrow{v}||^2 &= (\overrightarrow{u} + \overrightarrow{v})\bullet(\overrightarrow{u} + \overrightarrow{v}) - (\overrightarrow{u} - \overrightarrow{v})\bullet(\overrightarrow{u} - \overrightarrow{v}) \\ &= \overrightarrow{u}\bullet\overrightarrow{u} + 2\overrightarrow{u}\bullet\overrightarrow{v} + \overrightarrow{v}\bullet\overrightarrow{v} - (\overrightarrow{u}\bullet\overrightarrow{u} - 2\overrightarrow{u}\bullet\overrightarrow{v} + \overrightarrow{v}\bullet\overrightarrow{v}) \\ &= 4\overrightarrow{u}\bullet\overrightarrow{v}, \end{aligned}$$

giving the result.

1.5.6 A parametric equation for L_1 is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b_1 \end{pmatrix} + t\begin{bmatrix} 1 \\ m_1 \end{bmatrix}.$$

A parametric equation for L_2 is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b_2 \end{pmatrix} + t\begin{bmatrix} 1 \\ m_2 \end{bmatrix}.$$

The lines are perpendicular if and only if, according to Corollary 22,

$$\begin{bmatrix} 1 \\ m_1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ m_2 \end{bmatrix} = 0 \iff 1 + m_1 m_2 = 0 \iff m_1 m_2 = -1.$$

1.5.7 The line L has a parametric equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Let L' have parametric equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} + t\begin{bmatrix} 1 \\ m \end{bmatrix}.$$

We need the angle between $\begin{bmatrix} 1 \\ m \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to be $\frac{\pi}{6}$ and so by Theorem 21,

$$1 - m = \sqrt{1 + m^2}\sqrt{2}\cos\frac{\pi}{6} \implies m = -2 \pm \sqrt{3}.$$

This gives two possible values for the slope of L'. Now, since L' must pass through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$y = (-2 \pm \sqrt{3})x + b \implies 2 = (-2 \pm \sqrt{3})(-1) + b \implies b = \pm\sqrt{3}$$

and the lines are $y = (-2 + \sqrt{3})x + \sqrt{3}$ and $y = (-2 - \sqrt{3})x - \sqrt{3}$, respectively.

1.5.8 We must prove that $\vec{a} \bullet \vec{w} = 0$. Using the distributive law for the dot product,

$$\begin{aligned}
\left(\vec{v} - \frac{\vec{v} \bullet \vec{w}}{\left|\left|\vec{w}\right|\right|^2}\vec{w}\right) \bullet \vec{w} &= \vec{v} \bullet \vec{w} - \frac{\vec{v} \bullet \vec{w}}{\left|\left|\vec{w}\right|\right|^2}\vec{w} \bullet \vec{w} \\
&= \vec{v} \bullet \vec{w} - \frac{\vec{v} \bullet \vec{w}}{\left|\left|\vec{w}\right|\right|^2}\left|\left|\vec{w}\right|\right|^2 \\
&= \vec{v} \bullet \vec{w} - \vec{v} \bullet \vec{w} \\
&= 0.
\end{aligned}$$

1.6.1 We have $y - x = 4t \implies t = \frac{y-x}{4}$ and so

$$x = \left(\frac{y-x}{4}\right)^3 - 2\left(\frac{y-x}{4}\right)$$

is the Cartesian equation sought.

1.6.2 Observe that for $t \neq \{0, -1\}$,

$$\frac{y}{x} = t \implies x = \frac{\left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^5} \implies x = \frac{y^2x^3}{x^5 + y^5} \implies x^5 + y^5 = x^2y^2.$$

If $t = 0$, then $x = 0, y = 0$ and our Cartesian equation agrees. What happens as $t \to -1$?

1.6.3

1. $ay - cx = ad - bc$, this is a straight line with positive slope.
2. $-1 \leq x \leq 1, y = 0$, this is the line segment on the plane joining $(-1, 0)$ to $(1, 0)$.
3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x > 0$. This is one branch of a hyperbola.
4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This is a hyperbola.

1.6.4 We may simply give the trivial parametrisation: $x = t$, $y = \log\cos t$, $0 \leq t \leq \frac{\pi}{3}$. Then

$$(\mathrm{d}x)^2 + (\mathrm{d}y)^2 = (1 + \tan^2 t)(\mathrm{d}t)^2 = \sec^2 t(\mathrm{d}t)^2.$$

Hence the arc length is

$$\int_0^{\pi/3} \sec t\mathrm{d}t = \log(2 + \sqrt{3}).$$

1.6.5 Observe that $y = 2x + 1$, so the trace is part of this line. Since in the interval $[0; 4\pi]$, $-1 \leq \sin t \leq 1$, we want the portion of the line $y = 2x + 1$ with $-1 \leq x \leq 1$ (and, thus $-1 \leq y \leq 3$). The curve starts at the middle point $(0, 1)$ (at $t = 0$), reaches the high point $(1, 3)$ at $t = \frac{\pi}{2}$, reaches its low point $(-1, 1)$ at $t = \frac{3\pi}{2}$, reaches its high point $(1, 3)$ again at $t = \frac{5\pi}{2}$, it goes to its low point $(-1, 1)$ at $t = \frac{7\pi}{2}$, and finishes in the middle point $(0, 1)$ when $t = 4\pi$.

1.6.6 First observe that $\sqrt{x} + \sqrt{y} = 1$ demands $x \in [0; 1]$ and $[0; 1]$. Again, one can give many parametrisations. One is $x = t^2$, $y = (1 - t)^2$, $t \in [0; 1]$. This gives

$$\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{4t^2 + (2 - 2t)^2}\ \mathrm{d}t = 2\sqrt{2t^2 - 2t + 1}\ \mathrm{d}t = \frac{2}{\sqrt{2}}\sqrt{4\left(t - \frac{1}{2}\right)^2 + 1}\ \mathrm{d}t.$$

To integrate

$$\frac{2}{\sqrt{2}}\int_0^1 \sqrt{4\left(t - \frac{1}{2}\right)^2 + 1}\ \mathrm{d}t,$$

we now use the trigonometric substitution

$$2\left(t - \frac{1}{2}\right) = \tan\theta \implies \mathrm{d}t = \frac{1}{2}\sec^2\theta\mathrm{d}\theta.$$

The integral thus becomes

$$\sqrt{2}\int_0^{\pi/4}\sec^3\theta \mathrm{d}\theta,$$

the famous secant cube integral, which is a standard example of integration by parts where you "solve" for the integral. (You write $\int \sec\theta \mathrm{d}\tan\theta = \tan\theta\sec\theta - \int \tan\theta \mathrm{d}\sec\theta$, etc.) I will simply quote it, as I assume most of you have seen it, and it appears in most Calculus texts:

$$\begin{aligned}\sqrt{2}\int_0^{\pi/4}\sec^3\theta \mathrm{d}\theta &= \sqrt{2}\left(\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\log|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}\\ &= \sqrt{2}\left(\frac{1}{2}\cdot\sqrt{2}+\frac{1}{2}\log(\sqrt{2}+1)\right)\\ &= \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)+1.\end{aligned}$$

1.6.7 First notice that $x = 1 \implies t^3 + 1 = 1 \implies t = 0$ and $x = 2 \implies t^3 + 1 = 2 \implies t = 1$. The area under the graph is

$$\int_{t=0}^{t=1} y\mathrm{d}x = \int_{t=0}^{t=1}(1-t^2)\mathrm{d}(t^3+1) = \int_{t=0}^{t=1} 3t^2(1-t^2)\mathrm{d}t = \frac{2}{5}.$$

1.6.8 Observe that

$$\mathrm{d}x = 6t\mathrm{d}t; \quad \mathrm{d}y = 6t^2\mathrm{d}t \implies \sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = 6t\sqrt{1+t^2}\ \mathrm{d}t,$$

and so the arc length is

$$\int_{t=0}^{t=1}\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = \int_{t=0}^{t=1} 6t\sqrt{1+t^2}\ \mathrm{d}t = 4\sqrt{2}-2.$$

1.6.9 Observe that

$$\mathrm{d}x = t\mathrm{d}t, \qquad \mathrm{d}y = \sqrt{2t+1}\mathrm{d}t \implies \sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = \sqrt{t^2+(\sqrt{2t+1})^2}\mathrm{d}t = \sqrt{t^2+2t+1}\mathrm{d}t = (t+1)\mathrm{d}t.$$

Hence, the arc length is

$$\int_{-1/2}^{1/2}(1+t)\mathrm{d}t = \left(t+\frac{t^2}{2}\right)\Big|_{-1/2}^{1/2} = 1.$$

1.6.10 Observe that the parametrisation traverses the curve once clockwise if $t \in [0; 2\pi]$. The area is given by

$$\begin{aligned}\frac{1}{2}\oint_\Gamma \det\begin{bmatrix} x & \mathrm{d}x \\ y & \mathrm{d}y\end{bmatrix} &= \frac{1}{2}\oint x\mathrm{d}y - y\mathrm{d}x\\ &= \frac{4}{2}\int_{\pi/2}^{0}(\sin^3 t(-\sin t(1+\sin^2 t)+2\sin t\cos^2 t)\\ &\quad -\cos t(1+\sin^2 t)(3\sin^2 t\cos t))\mathrm{d}t\\ &= 2\int_{\pi/2}^{0}(-3\sin^2 t+\sin^4 t)\mathrm{d}t\\ &= 2\int_{\pi/2}^{0}\left(-\frac{9}{8}+\cos 2t+\frac{1}{8}\cos 4t\right)\mathrm{d}t\\ &= \frac{9\pi}{8}.\end{aligned}$$

1.6.11 Using the quotient rule,

$$\mathrm{d}x = \frac{3(1+t^3)-3t^2(3t)}{(1+t^3)^2}\cdot\mathrm{d}t = \frac{3-6t^3}{(1+t^3)^2}\cdot\mathrm{d}t \implies y\mathrm{d}x = \frac{9t^2-18t^5}{(1+t^3)^3}\cdot\mathrm{d}t$$

and

$$\mathrm{d}y = \frac{6t(1+t^3)-3t^2(3t^2)}{(1+t^3)^2}\cdot\mathrm{d}t = \frac{6t-3t^4}{(1+t^3)^2}\cdot\mathrm{d}t \implies x\mathrm{d}y = \frac{18t^2-9t^5}{(1+t^3)^3}\cdot\mathrm{d}t$$

Hence

$$x\mathrm{d}y - y\mathrm{d}x = \frac{18t^2-9t^5}{(1+t^3)^3}\cdot\mathrm{d}t - \frac{9t^2-18t^5}{(1+t^3)^3}\cdot\mathrm{d}t = \frac{9t^2+9t^5}{(1+t^3)^3}\cdot\mathrm{d}t = \frac{9t^2(1+t^3)}{(1+t^3)^3}\cdot\mathrm{d}t = \frac{9t^2}{(1+t^3)^2}\cdot\mathrm{d}t.$$

Observe that when $t = 0$ then $x = y = 0$. As $t \to +\infty$, then $x \to 0$ and $y \to 0$. Hence to obtain the loop Using integration by substitution ($u = 1 + t^3$ and $\mathrm{d}u = 3t^2\mathrm{d}t$) , the area is given by

$$\frac{1}{2}\int_0^{+\infty}\frac{9t^2}{(1+t^3)^2}\cdot\mathrm{d}t = \frac{3}{2}\int_0^{+\infty}\frac{3t^2}{(1+t^3)^2}\mathrm{d}t = \frac{3}{2}\int_1^{+\infty}\frac{\mathrm{d}u}{u^2} = \frac{3}{2}.$$

☞ *A shorter way of obtaining $x\mathrm{d}y - y\mathrm{d}x$ would have been to argue that $x\mathrm{d}y - y\mathrm{d}x = x^2\mathrm{d}\left(\frac{y}{x}\right) = \frac{9t^2}{(1+t^3)^2}\mathrm{d}t$.*

1.6.12 See figure 1.58. Let θ be the angle (in radians) of rotation of the circle, and let C be the centre of the circle. At $\theta = 0$ the centre of the circle is at $(0, \rho)$, and $P = (0, \rho - d)$. Suppose the circle is displaced towards the right, making the point P to rotate an angle of θ radians. Then the centre of the circle has displaced $r\theta$ units horizontally, and so is now located at $(\rho\theta, \rho)$. The polar coordinates of the point P are $(d\sin\theta; d\cos\theta)$, in relation to the centre of the circle (notice that the circle moves clockwise). The point P has moved $x = \rho\theta - d\sin\theta$ horizontal units and $y = \rho - d\cos\theta$ units. This is the desired parametrisation.

1.6.14 We have

$$\mathrm{d}x = e^t(\cos t - \sin t)\mathrm{d}t,\ \mathrm{d}y = e^t(\sin t + \cos t)\mathrm{d}t \implies \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}\mathrm{d}t = \sqrt{2}e^t\mathrm{d}t.$$

The arc length is thus

$$\int_0^\pi \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{2}\int_0^\pi e^t\mathrm{d}t = \sqrt{2}(e^\pi - 1).$$

1.6.15 Choose coordinates so that the origin is at the position of the gun, the y-axis is vertical, and the airplane is on a point with coordinates (u, h) with $u \geq 0$.

If the gun were fired at $t = 0$, then

$$x = Vt\cos a; \quad y = Vt\sin a - \frac{gt^2}{2},$$

where a is the angle of elevation, t is the time and g is the acceleration due to gravity. Since we know that the shell strikes the plane, we must have

$$u = Vt\cos a; \quad h = Vt\sin a - \frac{gt^2}{2},$$

whence

$$u^2 + \left(h + \frac{1}{2}gt^2\right)^2 = V^2t^2,$$

and thus

$$\frac{g^2t^4}{4} + (gh - V^2)^2t^2 + h^2 + u^2 = 0.$$

The quadratic equation in t^2 has a real root if

$$(gh - V^2)^2 \geq g^2(h^2 + u^2) \implies g^2u^2 \leq V^2(V^2 - 2gh),$$

from where the assertion follows.

1.6.16 Suppose the parabolas have a point of contact $P = (4px_0^2, 4px_0)$. By symmetry, the vertex V of the rolling parabola is the reflexion of the origin about the line tangent to their point of contact. The slope of the tangent at P is $\frac{1}{2x_0}$, from where the equation of the tangent is

$$y = \frac{x}{2x_0} + 2px_0.$$

The line normal to this line and passing through the origin is hence

$$y = -2xx_0,$$

and so the lines intersect at

$$\left(-\frac{4px_0^2}{1+4x_0^2}, \frac{8px_0^3}{1+4x_0^2}\right),$$

from where

$$V = \left(-\frac{8px_0^2}{1+4x_0^2}, \frac{16px_0^3}{1+4x_0^2}\right) := (x(t), y(t)).$$

As $-2x_0x(t) = y(t)$, eliminating t yields

$$x = -\frac{2p\left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^2}$$

or

$$(x^2 + y^2)x + 2py^2 = 0,$$

giving the equation of the locus.

1.7.1 Observe that, in general,

$$\begin{aligned} \left|\left|\vec{a} - \vec{b}\right|\right|^2 + \left|\left|\vec{a} + \vec{b}\right|\right|^2 &= \left|\left|\vec{a}\right|\right|^2 - 2\vec{a}\bullet\vec{b} + \left|\left|\vec{b}\right|\right|^2 + \left|\left|\vec{a}\right|\right|^2 + 2\vec{a}\bullet\vec{b} + \left|\left|\vec{b}\right|\right|^2 \\ &= 2\left|\left|\vec{a}\right|\right|^2 + 2\left|\left|\vec{b}\right|\right|^2, \end{aligned}$$

whence

$$\left|\left|\vec{a} - \vec{b}\right|\right| = \sqrt{2\left|\left|\vec{a}\right|\right|^2 + 2\left|\left|\vec{b}\right|\right|^2 - \left|\left|\vec{a} + \vec{b}\right|\right|} = \sqrt{2(13)^2 + 2(19)^2 - (24)^2} = 22.$$

1.7.2 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.$

1.7.3 The vectorial form of the equation of the line is

$$\vec{r} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

Since the line follows the direction of $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, this means that $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ is normal to the plane, and thus the equation of the desired plane is

$$(x - 1) - 2(y - 1) - (z - 1) = 0.$$

1.7.4 Observe that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (as $0 = 2(0) = 3(0)$) is on the line, and hence on the plane. Thus the vector

$$\begin{bmatrix} 1 - 0 \\ -1 - 0 \\ -1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

lies on the plane. Now, if $x = 2y = 3z = t$, then $x = t, y = t/2, z = t/3$. Hence, the vectorial form of the equation of the line is

$$\vec{r} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}.$$

This means that $\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$ also lies on the plane, and thus

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -4/3 \\ 3/2 \end{bmatrix}$$

is normal to the plane. The desired equation is thus

$$\frac{1}{6}x - \frac{4}{3}y + \frac{3}{2}z = 0.$$

1.7.5 The set B can be decomposed into the following subsets:

❶ The set A itself, of volume abc.

❷ Two $a \times b \times 1$ bricks, two $b \times c \times 1$ bricks, and two $c \times a \times 1$ bricks,

❸ Four quarter-cylinders of length a and radius 1, four quarter-cylinders of length b and radius 1, and four quarter-cylinders of length c and radius 1, ❹ Eight eighth-of-spheres of radius 1.

Thus the required formula for the volume is

$$abc + 2(ab + bc + ca) + \pi(a + b + c) + \frac{4\pi}{3}.$$

1.7.6 We have,

$$\left|\left|\vec{a} + \vec{b} + \vec{c}\right|\right|^2 = (\vec{a} + \vec{b} + \vec{c}) \bullet (\vec{a} + \vec{b} + \vec{c}) = \left|\left|\vec{a}\right|\right|^2 + \left|\left|\vec{b}\right|\right|^2 + \left|\left|\vec{c}\right|\right|^2 + 2(\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{c} + \vec{c}\bullet\vec{a}),$$

from where we deduce that

$$\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{c} + \vec{c}\bullet\vec{a} = \frac{-3^2 - 4^2 - 5^2}{2} = -25.$$

1.7.7 A vector normal to the plane is $\begin{bmatrix} a \\ a^2 \\ a^2 \end{bmatrix}$. The line sought has the same direction as this vector, thus the equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} a \\ a^2 \\ a^2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

1.7.8 Put $ax = by = cz = t$, so $x = t/a; y = t/b; z = t/c$. The parametric equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix}, \quad t \in \mathbb{R}.$$

Thus the vector $\begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix}$ is perpendicular to the plane. Therefore, the equation of the plane is

$$\begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix} \bullet \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

We may also write this as

$$bcx + cay + abz = ab + bc + ca.$$

1.7.9 The vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is perpendicular to the plane. Hence, the shortest distance from $(1, 2, 3)$ is obtained by the perpendicular line to the plane that pierces the plane, this perpendicular line to the plane has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \implies x = 1 + t, y = 2 - t, z = 3 + t.$$

The intersection of the line and the plane occurs when

$$1 + t - (2 - t) + (3 + t) = 1 \implies t = -\frac{1}{3}.$$

The closest point on the plane to $(1, 2, 3)$ is therefore $\left(\frac{2}{3}, \frac{7}{3}, \frac{8}{3}\right)$, and the distance sought is

$$\sqrt{(1 - \frac{2}{3})^2 + (2 - \frac{7}{3})^2 + (3 - \frac{8}{3})^2} = \frac{\sqrt{3}}{3}.$$

1.7.10 If the lines intersected, there would be a value t' for which

$$\begin{pmatrix}1\\1\\1\end{pmatrix}+t'\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}+t'\begin{bmatrix}2\\-1\\1\end{bmatrix}\implies\begin{bmatrix}1-0\\1-0\\1-1\end{bmatrix}=t'\begin{bmatrix}2-2\\-1-1\\1-1\end{bmatrix}\implies\begin{bmatrix}1\\1\\0\end{bmatrix}=t'\begin{bmatrix}0\\-2\\0\end{bmatrix},$$

which is clearly impossible, and so the lines are skew. Let θ be the angle between them. Then

$$\cos\theta=\frac{2\cdot 2+1\cdot(-1)+1\cdot 1}{\sqrt{(2)^2+(1)^2+(1)^2}\sqrt{(2)^2+(-1)^2+(1)^2}}=\frac{4}{\sqrt{6}\sqrt{6}}\implies\theta=\arccos\left(\frac{2}{3}\right).$$

1.7.11 Observe the CBS Inequality in $\mathbb{R}^3$ given the vectors $\overrightarrow{x}=(x_1,x_2,x_3)$, $\overrightarrow{y}=(y_1,y_2,y_3)$ let θ be the angle between them. Then

$$\overrightarrow{x}\bullet\overrightarrow{y}=\left|\left|\overrightarrow{x}\right|\right|\left|\left|\overrightarrow{y}\right|\right|\cos\theta\implies|x_1y_1+x_2y_2+x_3y_3|\leq\sqrt{x_1^2+x_2^2+x_3^2}\sqrt{y_1^2+y_2^2+y_3^2}.$$

Now take $x_1=a^2, x_2=b^2, x_3=c^2$ and $y_1=y_2=y_3=1$. This gives (since squares are positive, we don't need the absolute values)

$$|x_1y_1+x_2y_2+x_3y_3|\leq\sqrt{x_1^2+x_2^2+x_3^2}\sqrt{y_1^2+y_2^2+y_3^2}\implies(a^2+b^2+c^2)^2\leq(a^4+b^4+c^4)(3),$$

which proves the claim at once.

1.7.12 First observe that

$$\begin{aligned}S(a,b,c)&=\sqrt{s(s-a)(s-b)(s-c)}\\&=\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}\\&=\frac{1}{4}\sqrt{(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)}\end{aligned}$$

Hence

$$\frac{S(a,b,c)}{a^2+b^2+c^2}=\frac{1}{4}\sqrt{1-2\frac{a^4+b^4+c^4}{(a^2+b^2+c^2)^2}},$$

and thus maximising f is equivalent to minimising $2\frac{a^4+b^4+c^4}{(a^2+b^2+c^2)^2}$. From problem 1.7.11,

$$\frac{a^4+b^4+c^4}{(a^2+b^2+c^2)^2}\geq\frac{1}{3},$$

which in turn gives

$$\frac{S(a,b,c)}{a^2+b^2+c^2}\leq\frac{1}{4}\sqrt{1-\frac{2}{3}}=\frac{1}{4\sqrt{3}}=\frac{\sqrt{3}}{12},$$

the desired maximum.

1.7.15 $x+y+z=1$. $\frac{1}{6}$.

1.7.16 Assume contrariwise that $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ are three unit vectors in $\mathbb{R}^3$ such that the angle between any two of them is $>\frac{2\pi}{3}$. Then $\overrightarrow{a}\bullet\overrightarrow{b}<-\frac{1}{2}$, $\overrightarrow{b}\bullet\overrightarrow{c}<-\frac{1}{2}$, and $\overrightarrow{c}\bullet\overrightarrow{a}<-\frac{1}{2}$. Thus

$$\begin{aligned}\left|\left|\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c}\right|\right|^2&=\left|\left|\overrightarrow{a}\right|\right|^2+\left|\left|\overrightarrow{b}\right|\right|^2+\left|\left|\overrightarrow{c}\right|\right|^2\\&\quad+2\overrightarrow{a}\bullet\overrightarrow{b}+2\overrightarrow{b}\bullet\overrightarrow{c}+2\overrightarrow{c}\bullet\overrightarrow{a}\\&<1+1+1-1-1-1\\&=0,\end{aligned}$$

which is impossible, since a norm of vectors is always ≥ 0.

1.7.17 Let $\mathbf{proj}_{\mathbf{s}}^{\mathbf{t}} = \frac{\overrightarrow{\mathbf{t}} \bullet \overrightarrow{\mathbf{s}}}{(||s||)^2}\overrightarrow{\mathbf{s}}$ be the projection of $\overrightarrow{\mathbf{t}}$ over $\overrightarrow{\mathbf{s}} \neq \overrightarrow{\mathbf{0}}$. Let $\mathbf{x}_0$ be the point on the plane that is nearest to $\mathbf{b}$. Then $\overrightarrow{\mathbf{bx}_0} = \overrightarrow{\mathbf{x}_0} - \overrightarrow{\mathbf{b}}$ is orthogonal to the plane, and the distance we seek is

$$||\mathbf{proj}_{\mathbf{n}}^{\overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{b}}}|| = \left|\left|\frac{(\overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{b}})\bullet\overrightarrow{\mathbf{n}}}{\left|\left|\overrightarrow{\mathbf{n}}\right|\right|^2}\overrightarrow{\mathbf{n}}\right|\right| = \frac{|(\overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{b}})\bullet\overrightarrow{\mathbf{n}}|}{\left|\left|\overrightarrow{\mathbf{n}}\right|\right|}.$$

Since R_0 is on the plane, $\overrightarrow{\mathbf{r}_0}\bullet\overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{a}}\bullet\overrightarrow{\mathbf{n}}$, and so

$$||\mathbf{proj}_{\mathbf{n}}^{\overrightarrow{\mathbf{r}_0} - \overrightarrow{\mathbf{b}}}|| = \frac{|\overrightarrow{\mathbf{r}_0}\bullet\overrightarrow{\mathbf{n}} - \overrightarrow{\mathbf{b}}\bullet\overrightarrow{\mathbf{n}}|}{\left|\left|\overrightarrow{\mathbf{n}}\right|\right|} = \frac{|\overrightarrow{\mathbf{a}}\bullet\overrightarrow{\mathbf{n}} - \overrightarrow{\mathbf{b}}\bullet\overrightarrow{\mathbf{n}}|}{\left|\left|\overrightarrow{\mathbf{n}}\right|\right|} = \frac{|(\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}})\bullet\overrightarrow{\mathbf{n}}|}{\left|\left|\overrightarrow{\mathbf{n}}\right|\right|},$$

as we wanted to shew.

1.7.18 There are 7 vertices ($V_0 = (0, 0, 0)$, $V_1 = (11, 0, 0)$, $V_2 = (0, 9, 0)$, $V_3 = (0, 0, 8)$, $V_4 = (0, 3, 8)$, $V_5 = (9, 0, 2)$, $V_6 = (4, 7, 0)$) and 11 edges (V_0V_1, V_0V_2, V_0V_3, V_1V_5, V_1V_6, V_2V_4, V_2V_6, V_3V_4, V_3V_5, V_4V_5, and V_4V_6).

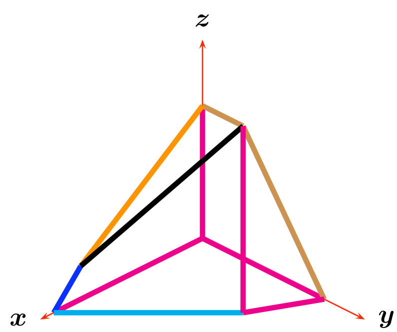

Figure A.11: Problem 1.7.18.

1.8.2 We have $\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}} = -\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}}$ by letting $\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{x}}$ in **??**. Thus $2\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$ and hence $\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$.

1.8.3 One has

$$(\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{a}}) \times (\overrightarrow{\mathbf{c}} - \overrightarrow{\mathbf{a}}) = \overrightarrow{\mathbf{0}} \implies \overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{c}} + \overrightarrow{\mathbf{c}} \times \overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{0}}$$

This gives

$$\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{c}} = -(\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{c}} \times \overrightarrow{\mathbf{a}}) = -(\overrightarrow{\mathbf{j}} + \overrightarrow{\mathbf{k}} + \overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{j}}) = -\overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{k}}.$$

1.8.4 The vector $\begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ is normal to the plane. The plane has thus equation

$$\begin{bmatrix}x\\y+1\\z-2\end{bmatrix} \bullet \begin{bmatrix}-1\\1\\0\end{bmatrix} = 0 \implies -x + y + 1 = 0 \implies x - y = 1,$$

as obtained before.

1.8.5 The vectors

$$\begin{bmatrix}a-(-a)\\0-1\\a-0\end{bmatrix} = \begin{bmatrix}2a\\-1\\a\end{bmatrix}$$

and

$$\begin{bmatrix}0-(-a)\\1-1\\2a-0\end{bmatrix} = \begin{bmatrix}a\\0\\2a\end{bmatrix}$$

lie on the plane. A vector normal to the plane is

$$\begin{bmatrix} 2a \\ -1 \\ a \end{bmatrix} \times \begin{bmatrix} a \\ 0 \\ 2a \end{bmatrix} = \begin{bmatrix} -2a \\ -3a^2 \\ a \end{bmatrix}.$$

The equation of the plane is thus given by

$$\begin{bmatrix} -2a \\ -3a^2 \\ a \end{bmatrix} \bullet \begin{bmatrix} x-a \\ y-0 \\ z-a \end{bmatrix} = 0,$$

that is,

$$2ax + 3a^2y - az = a^2.$$

1.8.6 Either of $\dfrac{\overrightarrow{v} \times \overrightarrow{w}}{||\overrightarrow{v} \times \overrightarrow{w}||}$ or $-\dfrac{\overrightarrow{v} \times \overrightarrow{w}}{||\overrightarrow{v} \times \overrightarrow{w}||}$ will do. Now

$$\begin{aligned} \overrightarrow{v} \times \overrightarrow{w} &= (-a\overrightarrow{j} + a\overrightarrow{k}) \times (\overrightarrow{i} + a\overrightarrow{j}) \\ &= -a(\overrightarrow{j} \times \overrightarrow{i}) - a^2(\overrightarrow{j} \times \overrightarrow{j}) + a(\overrightarrow{k} \times \overrightarrow{i}) + a^2(\overrightarrow{k} \times \overrightarrow{j}) \\ &= a\overrightarrow{k} + a\overrightarrow{j} - a^2\overrightarrow{i} \\ &= \begin{pmatrix} -a^2 \\ a \\ a \end{pmatrix}, \end{aligned}$$

and $||\overrightarrow{v} \times \overrightarrow{w}|| = \sqrt{a^4 + a^2 + a^2} = \sqrt{2a^2 + a^4}$. Hence we may take either

$$\frac{1}{\sqrt{2a^2 + a^4}} \begin{pmatrix} -a^2 \\ a \\ a \end{pmatrix}$$

or

$$-\frac{1}{\sqrt{2a^2 + a^4}} \begin{pmatrix} -a^2 \\ a \\ a \end{pmatrix}.$$

1.8.7 From Theorem **??** we have

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a} \bullet \overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a} \bullet \overrightarrow{b})\overrightarrow{c},$$

$$\overrightarrow{b} \times (\overrightarrow{c} \times \overrightarrow{a}) = (\overrightarrow{b} \bullet \overrightarrow{a})\overrightarrow{c} - (\overrightarrow{b} \bullet \overrightarrow{c})\overrightarrow{a},$$

$$\overrightarrow{c} \times (\overrightarrow{a} \times \overrightarrow{b}) = (\overrightarrow{c} \bullet \overrightarrow{b})\overrightarrow{a} - (\overrightarrow{c} \bullet \overrightarrow{a})\overrightarrow{b},$$

and adding yields the result.

1.8.8 By Lagrange's Identity,

$$||\overrightarrow{x} \times \overrightarrow{i}||^2 = \left|\left|\overrightarrow{x}\right|\right|^2 \left|\left|\overrightarrow{i}\right|\right|^2 - (\overrightarrow{x} \bullet \overrightarrow{i})^2 = 1 - (\overrightarrow{x} \bullet \overrightarrow{i})^2,$$

$$||\overrightarrow{x} \times \overrightarrow{k}||^2 = \left|\left|\overrightarrow{x}\right|\right|^2 \left|\left|\overrightarrow{j}\right|\right|^2 - (\overrightarrow{x} \bullet \overrightarrow{j})^2 = 1 - (\overrightarrow{x} \bullet \overrightarrow{j})^2,$$

$$||\overrightarrow{x} \times \overrightarrow{j}||^2 = \left|\left|\overrightarrow{x}\right|\right|^2 \left|\left|\overrightarrow{k}\right|\right|^2 - (\overrightarrow{x} \bullet \overrightarrow{k})^2 = 1 - (\overrightarrow{x} \bullet \overrightarrow{k})^2,$$

and since $(\overrightarrow{x} \bullet \overrightarrow{i})^2 + (\overrightarrow{x} \bullet \overrightarrow{j})^2 + (\overrightarrow{x} \bullet \overrightarrow{k})^2 = \left|\left|\overrightarrow{x}\right|\right|^2 = 1$, the desired sum equals $3 - 1 = 2$.

1.8.9

$$\overrightarrow{a} \times (\overrightarrow{x} \times \overrightarrow{b}) = \overrightarrow{b} \times (\overrightarrow{x} \times \overrightarrow{a}) \iff (\overrightarrow{a} \bullet \overrightarrow{b})\overrightarrow{x} - (\overrightarrow{a} \bullet \overrightarrow{x})\overrightarrow{b} = (\overrightarrow{b} \bullet \overrightarrow{a})\overrightarrow{x} - (\overrightarrow{b} \bullet \overrightarrow{x})\overrightarrow{a} \iff \overrightarrow{a} \bullet \overrightarrow{x} = \overrightarrow{b} \bullet \overrightarrow{x} = 0.$$

The answer is thus $\{\overrightarrow{x} : \overrightarrow{x} \in \mathbb{R}\overrightarrow{a} \times \overrightarrow{b}\}$.

1.8.12

$$\vec{x} = \frac{(\vec{a}\bullet\vec{b})\vec{a} + 6\vec{b} + 2\vec{a}\times\vec{c}}{12 + 2\left|\left|\vec{a}\right|\right|^2}$$

$$\vec{y} = \frac{(\vec{a}\bullet\vec{c})\vec{a} + 6\vec{c} + 3\vec{a}\times\vec{b}}{18 + 3\left|\left|\vec{a}\right|\right|^2}$$

1.8.13 First observe that

$$\vec{x}\bullet(\vec{y}\times\vec{z}) = (\vec{x}\times\vec{y})\bullet\vec{z}.$$

This is so because both sides give the volume of the parallelogram spanned by $\vec{x}$, $\vec{y}$, $\vec{z}$. Now, putting $\vec{x} = \vec{a}\times\vec{b}$, $\vec{y} = \vec{c}$ and $\vec{z} = \vec{d}$ we gather that

$$(\vec{a}\times\vec{b})\bullet(\vec{c}\times\vec{d}) \quad = \quad ((\vec{a}\times\vec{b})\times\vec{c})\bullet\vec{d}.$$

Now, again,

$$(\vec{a}\times\vec{b})\times\vec{c} = -\vec{c}\times(\vec{a}\times\vec{b}) = -((\vec{c}\bullet\vec{b})\vec{a} - (\vec{c}\bullet\vec{a})\vec{b}) = (\vec{c}\bullet\vec{a})\vec{b} - (\vec{c}\bullet\vec{b})\vec{a}.$$

This gives

$$((\vec{a}\times\vec{b})\times\vec{c})\bullet\vec{d} = ((\vec{c}\bullet\vec{a})\vec{b} - (\vec{c}\bullet\vec{b})\vec{a})\bullet\vec{d} = (\vec{c}\bullet\vec{a})(\vec{b}\bullet\vec{d}) - (\vec{c}\bullet\vec{b})(\vec{a}\bullet\vec{d}),$$

proving the identity.

1.8.14 By problem 1.8.13,

$$(\vec{x}\times\vec{y})\bullet(\vec{u}\times\vec{v}) = (\vec{x}\bullet\vec{u})\bullet(\vec{y}\bullet\vec{v}) - (\vec{x}\bullet\vec{v})\bullet(\vec{y}\bullet\vec{u}).$$

Using this three times:

$$\begin{array}{rcl}
(\vec{b}\times\vec{c})\bullet(\vec{a}\times\vec{d}) & = & (\vec{b}\bullet\vec{a})\bullet(\vec{c}\bullet\vec{d}) - (\vec{b}\bullet\vec{d})\bullet(\vec{c}\bullet\vec{d}) \\
(\vec{c}\times\vec{a})\bullet(\vec{b}\times\vec{d}) & = & (\vec{c}\bullet\vec{b})\bullet(\vec{a}\bullet\vec{d}) - (\vec{c}\bullet\vec{d})\bullet(\vec{a}\bullet\vec{b}) \\
(\vec{a}\times\vec{b})\bullet(\vec{c}\times\vec{d}) & = & (\vec{a}\bullet\vec{c})\bullet(\vec{b}\bullet\vec{d}) - (\vec{a}\bullet\vec{d})\bullet(\vec{b}\bullet\vec{c})
\end{array}$$

Adding these three equalities, and using the fact that the dot product is commutative, we see that all the terms on the dextral side cancel out and we obtain $\mathbf{0}$, as required.

1.8.15

1. We have

$$\overrightarrow{CA} = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}, \quad \overrightarrow{CB} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} \implies \overrightarrow{CA}\times\overrightarrow{CB} = (6\vec{i} - 3\vec{k})\times(4\vec{j} - 3\vec{k}) = 24\vec{k} + 18\vec{j} + 12\vec{i} = \begin{bmatrix} 12 \\ 18 \\ 24 \end{bmatrix}.$$

2. We have

$$\left|\left|\overrightarrow{CA}\times\overrightarrow{CB}\right|\right| = \sqrt{12^2 + 18^2 + 24^2} = \sqrt{1044} = 6\sqrt{29}.$$

3. The desired line has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + t\begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix} \implies x = 6t, \quad y = 0, \quad z = 3 - 3t.$$

4. The desired line has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \implies x = 3, \quad y = 0, \quad z = -3s.$$

5. From the preceding items, the line L_{CA} is $x = 6t,\ y = 0,\ z = 3 - 3t$ and the line L_{DE} is $x = 3,\ y = 0,\ z = -3s$. If the line intersect then $6t = 3,\ 0 = 0,\ 3 - 3t = -3s$ gives $t = \frac{1}{2}$ and $s = -\frac{1}{2}$. The point of intersection is thus $\left(3, 0, \frac{3}{2}\right)$. item The area is

$$\frac{1}{2}\left|\left|\overrightarrow{\mathrm{CA}} \times \overrightarrow{\mathrm{CB}}\right|\right| = \frac{1}{2} \cdot 6\sqrt{29} = 3\sqrt{29}.$$

6. Observe that

$$P = \begin{pmatrix} 3 \\ 0 \\ z \end{pmatrix}, \quad Q = \begin{pmatrix} 3 \\ y \\ 0 \end{pmatrix}, \quad R = \begin{pmatrix} x \\ 3 \\ 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 \\ 3 \\ z \end{pmatrix}.$$

Since all this points lie on the plane $2x + 3y + 4z = 12$, we find

$$2(3) + 3(0) + 4z = 12 \implies P = \begin{pmatrix} 3 \\ 0 \\ \frac{3}{2} \end{pmatrix},$$

$$2(3) + 3y + 4(0) = 12 \implies Q = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix},$$

$$2x + 3(3) + 4(0) = 12 \implies R = \begin{pmatrix} \frac{3}{2} \\ 3 \\ 0 \end{pmatrix},$$

$$2(0) + 3(3) + 4z = 12 \implies S = \begin{pmatrix} 0 \\ 3 \\ \frac{3}{4} \end{pmatrix}.$$

7. A possible way is to decompose the pentagon into three triangles, say $\triangle CPQ$, $\triangle CQR$ and $\triangle CRS$ and find their areas. Another way would be to subtract from the area of $\triangle ABC$ the areas of $\triangle APQ$ and $\triangle RSB$. I will follow the second approach. Let $[\triangle APQ]$, $[\triangle RSB]$ denote the areas of $\triangle APQ$ and $\triangle RSB$ respectively. Then

$$[\triangle APQ] = \frac{1}{2}\left|\left|\overrightarrow{\mathrm{PA}} \times \overrightarrow{\mathrm{PQ}}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} 3 \\ 0 \\ \frac{3}{2} \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ -\frac{3}{2} \end{bmatrix}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} -3 \\ \frac{9}{2} \\ 6 \end{bmatrix}\right|\right| = \frac{1}{2}\sqrt{9 + \frac{81}{4} + 36} = \frac{3}{4}\sqrt{29},$$

$$[\triangle RSB] = \frac{1}{2}\left|\left|\overrightarrow{\mathrm{SR}} \times \overrightarrow{\mathrm{SB}}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -\frac{3}{4} \end{bmatrix}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} \frac{3}{4} \\ \frac{9}{8} \\ \frac{3}{2} \end{bmatrix}\right|\right| = \frac{1}{2}\sqrt{\frac{9}{16} + \frac{81}{64} + \frac{9}{4}} = \frac{3}{16}\sqrt{29}.$$

Hence the area of the pentagon is

$$3\sqrt{29} - \frac{3}{4}\sqrt{29} - \frac{3}{16}\sqrt{29} = \frac{33}{16}\sqrt{29}.$$

1.9.2 The assertion is trivial for $n = 1$. Assume its truth for $n - 1$, that is, assume $A^{n-1} = 3^{n-2}A$. Observe that

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3A.$$

Now

$$A^n = AA^{n-1} = A(3^{n-2}A) = 3^{n-2}A^2 = 3^{n-2}3A = 3^{n-1}A,$$

and so the assertion is proved by induction.

1.9.3 First, we will prove that

$$A^2 = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 0 & 1 & 2 & 3 & \dots & n-2 & n-1 \\ 0 & 0 & 1 & 2 & \dots & n-3 & n-2 \\ \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Observe that $A = [a_{ij}]$, where $a_{ij} = 1$ for $i \leq j$ and $a_{ij} = 0$ for $i > j$.

Put $A^2 = [b_{ij}]$. Assume first that $i \leq j$. Then

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=i}^{j} 1 = j - i + 1.$$

Assume now that $i > j$. Then

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} 0 = 0,$$

proving the first statement. Now, we will prove that

$$A^3 = \begin{bmatrix} 1 & 3 & 6 & 10 & \dots & \frac{(n-1)n}{2} & \frac{n(n+1)}{2} \\ 0 & 1 & 3 & 6 & \dots & \frac{(n-2)(n-1)}{2} & \frac{(n-1)n}{2} \\ 0 & 0 & 1 & 3 & \dots & \frac{(n-3)(n-2)}{2} & \frac{(n-2)(n-1)}{2} \\ \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

For the second part, you need to know how to sum arithmetic progressions. In our case, we need to know how to sum (assume $i \leq j$),

$$S_1 = \sum_{k=i}^{j} a, \quad S_2 = \sum_{k=i}^{j} k.$$

The first sum is trivial: there are $j - i + 1$ integers in the interval $[i; j]$, and hence

$$S_1 = \sum_{k=i}^{j} a = S_1 = a\sum_{k=i}^{j} 1 = a(j - i + 1).$$

For the second sum, we use Gauß trick: summing the sum forwards is the same as summing the sum backwards, and so, adding the first two rows below,

$$\begin{array}{lcccccccccccc} S_2 & = & i & + & i+1 & + & i+2 & + & \cdots & + & j-1 & + & j \\ S_2 & = & j & + & j-1 & + & j-2 & + & \cdots & + & i-1 & + & i \\ \hline 2S_2 & = & (i+j) & + & (i+j) & + & (i+j) & + & \cdots & + & (i+j) & + & (i+j) \\ \hline 2S_2 & = & (i+j)(j-i+1) & & & & & & \cdots & & & & \end{array},$$

which gives $S_2 = \dfrac{(i+j)(j-i+1)}{2}$.

Put now $A^3 = [c_{ij}]$. Assume first that $i \leq j$. Since $A^3 = A^2A$,

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} b_{ik}a_{kj} \\ &= \sum_{k=i}^{j} (k - i + 1) \\ &= \sum_{k=i}^{j} k - \sum_{k=i}^{j} i + \sum_{k=i}^{j} 1 \\ &= \frac{(j+i)(j-i+1)}{2} - i(j - i + 1) + (j - i + 1) \\ &= \frac{(j-i+1)(j-i+2)}{2}. \end{aligned}$$

Assume now that $i > j$. Then

$$c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{n} 0 = 0.$$

This finishes the proof.

1.9.4 For the first part, observe that

$$\begin{aligned}
m(a)m(b) &= \begin{bmatrix} 1 & 0 & a \\ -a & 1 & -\frac{a^2}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b \\ -b & 1 & -\frac{b^2}{2} \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & a+b \\ -a-b & 1 & -\frac{a^2}{2} - \frac{b^2}{2} + ab \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & a+b \\ -(a+b) & 1 & -\frac{(a+b)^2}{2} \\ 0 & 0 & 1 \end{bmatrix} \\
&= m(a+b)
\end{aligned}$$

For the second part, observe that using the preceding part of the problem,

$$m(a)m(-a) = m(a-a) = m(0) = \begin{bmatrix} 1 & 0 & 0 \\ -0 & 1 & -\frac{0^2}{2} \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3,$$

giving the result.

1.11.1 $\frac{a}{\sqrt{2}}, \frac{\pi}{4}$.

1.11.3 $\frac{a}{\sqrt{3}}$

1.12.1 Consider a right triangle $\triangle ABC$ rectangle at A with legs of length $CA = h$ and $AB = r$, as in figure A.12. The cone is generated when the triangle rotates about CA. The gyrating curve is the hypotenuse, whose centroid is its centre. The length of the generating curve is thus $\sqrt{r^2+h^2}$ and the length of curve described by the centre of gravity is $2\pi\left(\frac{r}{2}\right) = \pi r$. The lateral area is thus $\pi r\sqrt{r^2+h^2}$.

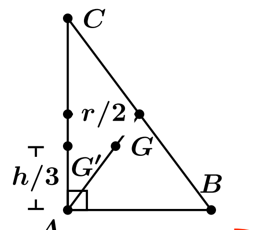

Figure A.12: Generating a cone.

To find the volume, we gyrate the whole right triangle, whose area is $\frac{rh}{2}$. We need to find the centroid of the triangle. But from example 17, we know that the centroid G of the triangle is is two thirds of the way from A to the midpoint of BC. If G' is the perpendicular projection of G onto $[CA]$, then this means that G' is at a vertical height of $\frac{h}{2} \cdot \frac{2}{3} = \frac{h}{3}$. By similar triangles $\frac{GG'}{r} = \frac{h/3}{h} \implies GG' = \frac{r}{3}$. Hence, the length of the curve described by the centre of gravity of the triangle is $\frac{2}{3}\pi r$. The volume of the cone is thus $\frac{2}{3}\pi r \cdot \frac{rh}{2} = \frac{\pi}{3}r^2h$.

1.14.1 Find a vector $\overrightarrow{a}$ mutually perpendicular to $\overrightarrow{V_1V_2}$ and $\overrightarrow{V_1V_3}$ and another vector and a vector $\overrightarrow{b}$ mutually perpendicular to $\overrightarrow{V_1V_3}$ and $\overrightarrow{V_1V_4}$. Then shew that $\cos\theta = \frac{1}{3}$, where θ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.

1.15.1 Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a point on S. If this point were on the xz plane, it would be on the ellipse, and its distance to the axis of rotation would be $|x| = \frac{1}{2}\sqrt{1-z^2}$. Anywhere else, the distance from $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the z-axis is the distance of this point to the point $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$: $\sqrt{x^2+y^2}$. This distance is the same as the length of the segment on the xz-plane going from the z-axis. We thus have

$$\sqrt{x^2+y^2} = \frac{1}{2}\sqrt{1-z^2},$$

or

$$4x^2 + 4y^2 + z^2 = 1.$$

1.15.2 Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a point on S. If this point were on the xy plane, it would be on the line, and its distance to the axis of rotation would be $|x| = \frac{1}{3}|1-4y|$. Anywhere else, the distance of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the axis of rotation is the same as the distance of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$, that is $\sqrt{x^2+z^2}$. We must have

$$\sqrt{x^2+z^2} = \frac{1}{3}|1-4y|,$$

which is to say

$$9x^2 + 9z^2 - 16y^2 + 8y - 1 = 0.$$

1.15.3 A spiral staircase.

1.15.4 A spiral staircase.

1.15.6 The planes $A : x + z = 0$ and $B : y = 0$ are secant. The surface has equation of the form $f(A, B) = e^{A^2+B^2} - A = 0$, and it is thus a cylinder. The directrix has direction $\overrightarrow{i} - \overrightarrow{k}$.

1.15.7 Rearranging,

$$(x^2+y^2+z^2)^2 - \frac{1}{2}((x+y+z)^2 - (x^2+y^2+z^2)) - 1 = 0,$$

and so we may take $A : x + y + z = 0, \Sigma : x^2 + y^2 + z^2 = 0$, shewing that the surface is of revolution. Its axis is the line in the direction $\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$.

1.15.8 Considering the planes $A: x-y=0, B: y-z=0$, the equation takes the form

$$f(A,B)=\frac{1}{A}+\frac{1}{B}-\frac{1}{A+B}-1=0,$$

thus the equation represents a cylinder. To find its directrix, we find the intersection of the planes $x=y$ and $y=z$. This gives $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The direction vector is thus $\overrightarrow{\mathrm{i}}+\overrightarrow{\mathrm{j}}+\overrightarrow{\mathrm{k}}$.

1.15.9 Rearranging,

$$(x+y+z)^2-(x^2+y^2+z^2)+2(x+y+z)+2=0,$$

so we may take $A: x+y+z=0, \Sigma: x^2+y^2+z^2=0$ as our plane and sphere. The axis of revolution is then in the direction of $\overrightarrow{\mathrm{i}}+\overrightarrow{\mathrm{j}}+\overrightarrow{\mathrm{k}}$.

1.15.10 After rearranging, we obtain

$$(z-1)^2-xy=0,$$

or

$$-\frac{x}{z-1}\frac{y}{z-1}+1=0.$$

Considering the planes

$$A: x=0,\ B: y=0,\ \ C: z=1,$$

we see that our surface is a cone, with apex at $(0,0,1)$.

1.15.11 The largest circle has radius b. Parallel cross sections of the ellipsoid are similar ellipses, hence we may increase the size of these by moving towards the centre of the ellipse. Every plane through the origin which makes a circular cross section must intersect the yz-plane, and the diameter of any such cross section must be a diameter of the ellipse $x=0, \frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Therefore, the radius of the circle is at most b. Arguing similarly on the xy-plane shews that the radius of the circle is at least b. To shew that circular cross section of radius b actually exist, one may verify that the two planes given by $a^2(b^2-c^2)z^2=c^2(a^2-b^2)x^2$ give circular cross sections of radius b.

1.15.12 Any hyperboloid oriented like the one on the figure has an equation of the form

$$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

When $z=0$ we must have

$$4x^2+y^2=1 \implies a=\frac{1}{2},\ b=1.$$

Thus

$$\frac{z^2}{c^2}=4x^2+y^2-1.$$

Hence, letting $z=\pm 2$,

$$\frac{4}{c^2}=4x^2+y^2-1 \implies \frac{1}{c^2}=x^2+\frac{y^2}{4}-\frac{1}{4}=1-\frac{1}{4}=\frac{3}{4},$$

since at $z=\pm 2$, $x^2+\frac{y^2}{4}=1$. The equation is thus

$$\frac{3z^2}{4}=4x^2+y^2-1.$$

1.16.1 The arc length element is

$$\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2}=\sqrt{4t^2+36t+81}\ \mathrm{d}t=(2t+9)\ \mathrm{d}t.$$

We need $t=1$ to $t=4$. The desired length is

$$\int_1^4 (2t+9)\ \mathrm{d}t=(t^2+9t)\Big|_1^4=(16+36)-(1+9)=42.$$

1.16.2 Observe that $z = 1 + x^2 + y^2$ is a paraboloid opening up, with vertex (lowest point) at $(0, 0, 1)$. On the other hand, $z = 3 - x^2 - y^2$ is another paraboloid opening down, with highest point at $(0, 0, 3)$. Adding the equations we obtain

$$2z = z + z = (1 + x^2 + y^2) + (3 - x^2 - y^2) = 4 \implies 2z = 4 \implies z = 2,$$

so they intersect at the plane $z = 2$. Subtracting the equations,

$$(1 + x^2 + y^2) - (3 - x^2 - y^2) = z - z = 0 \implies 2x^2 + 2y^2 - 2 = 0 \implies x^2 + y^2 = 1,$$

so they intersect at the circle $x^2 + y^2 = 1, z = 2$. Since they meet at the circle $x^2 + y^2 = 1$, we may parametrise this circle as $x = \cos t, y = \sin t$, $t \in [0; 2\pi]$, $z = 2$.

1.16.3 Let $\overrightarrow{r}(t)$ lie on the plane $ax + by + cz = d$. Then we must have

$$a\frac{t^4}{1+t^2} + b\frac{t^3}{1+t^2} + c\frac{t^2}{1+t^2} = d \implies (at^4 + bt^3 + ct^2) = d(1 + t^2) \implies at^4 + bt^3 + (c - d)t^2 - d = 0,$$

which means that if $\overrightarrow{r}(t)$ is on the plane $ax + by + cz = d$, then t must satisfy the quartic polynomial $p(t) = at^4 + bt^3 + (c - d)t^2 - d = 0$. Hence, the t_k are coplanar if and only if they are roots of $p(t)$. Since the coefficient of t in this polynomial is 0, then the sum of the roots of $p(t)$ taken three at a time is 0, that is,

$$t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 = 0 \implies \frac{t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4}{t_1t_2t_3t_4} = 0 \implies \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = 0,$$

as required.

1.16.4 Observe that in this problem you are *only parametrising the ellipsoid!* The tricky part is to figure out the bounds in your parameters so that *only* the part above the plane $x + y + z = 0$ is described. A common parametrisation found was:

$$x = \cos\theta\sin\phi, \quad y = 3\sin\theta\sin\phi, \quad z = \cos\phi.$$

The projection of the plane $x + y + z = 0$ onto the xy-plane is the line $y = -x$. To be "above" this line, the angle θ, measured from the positive x-axis needs to be in the interval $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$. Since $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is normal to the plane $x + y + z = 0$, the plane makes an angle of $\frac{\pi}{4}$ with the z-axis. Recall that ϕ is measured from $\phi = 0$ (positive z-axis) to $\phi = \pi$ (negative z-axis). Hence to be above the plane we need $0 \le \phi \le \frac{3\pi}{4}$.

1.16.6

1. $\begin{bmatrix}-2a\\a^2 - 1\\a^2 + 1\end{bmatrix}$
2. $x^2 + y^2 = c^2 + 1$
3. $\pi\int_0^1 (c^2 + 1)\mathrm{d}c = \frac{4\pi}{3}$

1.17.1

1. Put $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x-1} - x$. Clearly $f(1) = e^0 - 1 = 0$. Now,

$$f'(x) = e^{x-1} - 1,$$

$$f''(x) = e^{x-1}.$$

If $f'(x) = 0$ then $e^{x-1} = 1$ implying that $x = 1$. Thus f has a single minimum point at $x = 1$. Thus for all real numbers x

$$0 = f(1) \le f(x) = e^{x-1} - x,$$

which gives the desired result.

2. Easy Algebra!
3. Easy Algebra!

4. By the preceding results, we have

$$\begin{aligned} A_1 &\leq \exp(A_1 - 1), \\ A_2 &\leq \exp(A_2 - 1), \\ &\vdots \\ A_n &\leq \exp(A_n - 1). \end{aligned}$$

Since all the quantities involved are non-negative, we may multiply all these inequalities together, to obtain,

$$A_1 A_2 \cdots A_n \leq \exp(A_1 + A_2 + \cdots + A_n - n).$$

In view of the observations above, the preceding inequality is equivalent to

$$\frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n} \leq \exp(n - n) = e^0 = 1.$$

We deduce that

$$G_n \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n,$$

which is equivalent to

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now, for equality to occur, we need each of the inequalities $A_k \leq \exp(A_k - 1)$ to hold. This occurs, in view of the preceding lemma, if and only if $A_k = 1$, $\forall k$, which translates into $a_1 = a_2 = \ldots = a_n$. This completes the proof.

1.17.2 By CBS,

$$\begin{aligned} (x_1 + x_2 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right) &\geq \left(\sum_{i=1}^n \sqrt{x_i}\frac{1}{\sqrt{x_i}}\right)^2 \\ &= n^2. \end{aligned}$$

1.17.3 By CBS,

$$(a + b + c + d)^2 \leq (1 + 1 + 1 + 1)\left(a^2 + b^2 + c^2 + d^2\right) = 4\left(a^2 + b^2 + c^2 + d^2\right).$$

Hence,

$$(8 - e)^2 \leq 4\left(16 - e^2\right) \iff e\,(5e - 16) \leq 0 \iff 0 \leq e \leq \frac{16}{5}.$$

The maximum value $e = \dfrac{16}{5}$ is reached when $a = b = c = d = \dfrac{6}{5}$.

1.17.4 Observe that $96 \cdot 216 = 144^2$ and by CBS,

$$\sum_{k=1}^n a_k^2 \leq \left(\sum_{k=1}^n a_k^3\right)\left(\sum_{k=1}^n a_k\right).$$

As there is equality,

$$(a_1, a_2, \ldots, a_n) = t(a_1^3, a_2^3, \ldots, a_n^3)$$

for some real number t. Hence $a_1 = a_2 = \ldots = a_n = a$, from where $na = 96$, $na^2 = 144$ gives $a = \dfrac{3}{2}$ y $n = 32$.

1.17.5 Applying the AM-GM inequality, for $1, 2, \ldots, n$:

$$n!^{1/n} = (1 \cdot 2 \cdots n)^{1/n} < \frac{1 + 2 + \cdots + n}{n} = \frac{n+1}{2},$$

with strict inequality for $n > 1$.

1.17.6 If $x \in [-a; a]$, then $a + x \geq 0$ and $a - x \geq 0$, and thus we may use AM-GM with $n = 8$, $a_1 = a_2 = \cdots = a_5 = \dfrac{a+x}{5}$ and $a_6 = a_7 = a_8 = \dfrac{a-x}{3}$. We deduce that

$$\left(\frac{a+x}{5}\right)^5\left(\frac{a-x}{3}\right)^3 \leq \left(\frac{5\left(\frac{a+x}{5}\right) + 3\left(\frac{a-x}{3}\right)}{8}\right)^8 = \left(\frac{a}{4}\right)^8,$$

from where

$$f(x) \leq \frac{5^5 3^3 a^8}{4^8},$$

with equality if and only if $\dfrac{a+x}{5} = \dfrac{a-x}{3}$.

1.17.7 Applying AM-GM to the set of $n+1$ numbers

$$1, 1+\frac{1}{n}, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n},$$

has arithmetic mean

$$1+\frac{1}{n+1}$$

and geometric mean

$$\left(1+\frac{1}{n}\right)^{n/(n+1)}.$$

Therefore,

$$1+\frac{1}{n+1} > \left(1+\frac{1}{n}\right)^{n/(n+1)},$$

that is

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n},$$

which means

$$x_{n+1} > x_n,$$

giving the assertion.

2.1.2 Since polynomials are continuous functions and the image of a connected set is connected for a continuous function, the image must be an interval of some sort. If the image were a finite interval, then $f(x, kx)$ would be bounded for every constant k, and so the image would just be the point $f(0,0)$. The possibilities are thus

1. a single point (take for example, $p(x,y) = 0$),
2. a semi-infinite interval with an endpoint (take for example $p(x,y) = x^2$ whose image is $[0; +\infty[$),
3. a semi-infinite interval with no endpoint (take for example $p(x,y) = (xy-1)^2 + x^2$ whose image is $]0; +\infty[$),
4. all real numbers (take for example $p(x,y) = x$).

2.3.4 0

2.3.5 2

2.3.6 $c = 0$.

2.3.7 0

2.3.10 By AM-GM,

$$\frac{x^2y^2z^2}{x^2+y^2+z^2} \leq \frac{(x^2+y^2+z^2)^3}{27(x^2+y^2+z^2)} = \frac{(x^2+y^2+z^2)^2}{27} \to 0$$

as $(x,y,z) \to (0,0,0)$.

2.4.1 We have

$$\begin{aligned} F(\overrightarrow{x}+\overrightarrow{h}) - F(\overrightarrow{x}) &= (\overrightarrow{x}+\overrightarrow{h}) \times L(\overrightarrow{x}+\overrightarrow{h}) - \overrightarrow{x} \times L(\overrightarrow{x}) \\ &= (\overrightarrow{x}+\overrightarrow{h}) \times (L(\overrightarrow{x}) + L(\overrightarrow{h})) - \overrightarrow{x} \times L(\overrightarrow{x}) \\ &= \overrightarrow{x} \times L(\overrightarrow{h}) + \overrightarrow{h} \times L(\overrightarrow{x}) + \overrightarrow{h} \times L(\overrightarrow{h}) \end{aligned}$$

Now, we will prove that $||\overrightarrow{h} \times L(\overrightarrow{h})|| = o\left(\left|\left|\overrightarrow{h}\right|\right|\right)$ as $\overrightarrow{h} \to \overrightarrow{0}$. For let

$$\overrightarrow{h} = \sum_{k=1}^{n} h_k \overrightarrow{e}_k,$$

where the $\overrightarrow{e}_k$ are the standard basis for $\mathbb{R}^n$. Then

$$L(\overrightarrow{h}) = \sum_{k=1}^{n} h_k L(\overrightarrow{e}_k),$$

and hence by the triangle inequality, and by the Cauchy-Bunyakovsky-Schwarz inequality,

$$\begin{aligned} ||L(\overrightarrow{h})|| &\leq \sum_{k=1}^{n} |h_k|||L(\overrightarrow{e}_k)|| \\ &\leq \left(\sum_{k=1}^{n} |h_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} ||L(\overrightarrow{e}_k)||^2\right)^{1/2} \\ &= \left|\left|\overrightarrow{h}\right|\right|(\sum_{k=1}^{n} ||L(\overrightarrow{e}_k)||^2)^{1/2}, \end{aligned}$$

whence, again by the Cauchy-Bunyakovsky-Schwarz Inequality,

$$||\overrightarrow{h} \times L(\overrightarrow{h})|| \leq ||\overrightarrow{h}||||L(\overrightarrow{h})| \leq ||\overrightarrow{h}||^2|||L(\overrightarrow{e}_k)||^2)^{1/2} = o\left(||\overrightarrow{h}||\right),$$

as it was to be shewn.

2.4.2 Assume that $\overrightarrow{x} \neq \overrightarrow{0}$. We use the fact that $(1+t)^{1/2} = 1 + \frac{t}{2} + o(t)$ as $t \to 0$. We have

$$\begin{aligned} f(\overrightarrow{x} + \overrightarrow{h}) - f(\overrightarrow{x}) &= ||\overrightarrow{x} + \overrightarrow{h}|| - \left|\left|\overrightarrow{x}\right|\right| \\ &= \sqrt{(\overrightarrow{x} + \overrightarrow{h})\bullet(\overrightarrow{x} + \overrightarrow{h})} - \left|\left|\overrightarrow{x}\right|\right| \\ &= \sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} - \left|\left|\overrightarrow{x}\right|\right| \\ &= \frac{2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2}{\sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} + \left|\left|\overrightarrow{x}\right|\right|}. \end{aligned}$$

As $\overrightarrow{h} \to \overrightarrow{0}$,

$$\sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} + \left|\left|\overrightarrow{x}\right|\right| \to 2\left|\left|\overrightarrow{x}\right|\right|.$$

Since $\left|\left|\overrightarrow{h}\right|\right|^2 = o\left(\left|\left|\overrightarrow{h}\right|\right|\right)$ as $\overrightarrow{h} \to \overrightarrow{0}$, we have

$$\frac{2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2}{\sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} + \left|\left|\overrightarrow{x}\right|\right|} \to \frac{\overrightarrow{x}\bullet\overrightarrow{h}}{\left|\left|\overrightarrow{h}\right|\right|} + o\left(\left|\left|\overrightarrow{h}\right|\right|\right),$$

proving the first assertion.

To prove the second assertion, assume that there is a linear transformation $\mathscr{D}_0(f) = L$, $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$||f(\overrightarrow{0} + \overrightarrow{h}) - f(\overrightarrow{0}) - L(\overrightarrow{h})|| = o\left(\left|\left|\overrightarrow{h}\right|\right|\right),$$

as $\left|\left|\overrightarrow{h}\right|\right| \to 0$. Recall that by theorem **??**, $L(\overrightarrow{0}) = \overrightarrow{0}$, and so by example 169, $\mathscr{D}_0(L)(\overrightarrow{0}) = L(\overrightarrow{0}) = \overrightarrow{0}$. This implies that $\frac{L(\overrightarrow{h})}{\left|\left|\overrightarrow{h}\right|\right|} \to \mathscr{D}_0(L)(\overrightarrow{0}) = \overrightarrow{0}$, as $\left|\left|\overrightarrow{h}\right|\right| \to 0$. Since $f(\overrightarrow{0}) = ||0|| = 0$, $f(\overrightarrow{h}) = \left|\left|\overrightarrow{h}\right|\right|$ this would imply that

$$\left|\left|\left|\left|\overrightarrow{h}\right|\right| - L(\overrightarrow{h})\right|\right| = o\left(\left|\left|\overrightarrow{h}\right|\right|\right),$$

or

$$\left|\left|1 - \frac{L(\overrightarrow{h})}{\left|\left|\overrightarrow{h}\right|\right|}\right|\right| = o(1).$$

But the sinistral side $\to 1$ as $\overrightarrow{h} \to \overrightarrow{0}$, and the dextral side $\to 0$ as $\overrightarrow{h} \to \overrightarrow{0}$. This is a contradiction, and so, such linear transformation L does not exist at the point $\overrightarrow{0}$.

2.5.2 Observe that

$$f(x,y)=\begin{cases} x & \text{if } x\leq y^2\\ y^2 & \text{if } x>y^2\end{cases}$$

Hence

$$\frac{\partial}{\partial x}f(x,y)=\begin{cases} 1 & \text{if } x>y^2\\ 0 & \text{if } x>y^2\end{cases}$$

and

$$\frac{\partial}{\partial y}f(x,y)=\begin{cases} 0 & \text{if } x>y^2\\ 2y & \text{if } x>y^2\end{cases}$$

2.5.3 Observe that

$$g(1,0,1)=\begin{bmatrix}3\\0\end{bmatrix},\qquad f'(x,y)=\begin{bmatrix}y^2 & 2xy\\ 2xy & x^2\end{bmatrix},\qquad g'(x,y)=\begin{bmatrix}1 & -1 & 2\\ y & x & 0\end{bmatrix},$$

and hence

$$g'(1,0,1)=\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 0\end{bmatrix},\qquad f'(g(1,0,1))=f'(3,0)=\begin{bmatrix}0 & 0\\ 0 & 9\end{bmatrix}.$$

This gives, via the Chain-Rule,

$$(f\circ g)'(1,0,1)=f'(g(1,0,1))g'(1,0,1)=\begin{bmatrix}0 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 9 & 0\end{bmatrix}.$$

The composition $g\circ f$ is undefined. For, the output of f is $\mathbb{R}^2$, but the input of g is in $\mathbb{R}^3$.

2.5.4 Since $f(0,1)=\begin{bmatrix}0\\1\end{bmatrix}$, the Chain Rule gives

$$(g\circ f)'(0,1)=(g'(f(0,1)))(f'(0,1))=(g'(0,1))(f'(0,1))=\begin{bmatrix}1 & -1\\ 0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 0 & 0\\ 2 & 1\end{bmatrix}$$

2.5.5 We have

$$\frac{\partial f}{\partial x}(x,y,z)=2xyg(x^2y),$$

and

$$\frac{\partial f}{\partial y}(x,y,z)=x^2g(x^2y).$$

2.5.6 Differentiating both sides with respect to the parameter a, the integral is $\dfrac{1}{2a^3}\arctan\dfrac{b}{a}+\dfrac{b}{2a^2(a^2+b^2)}$

2.5.10 We have

$$\frac{\partial}{\partial x}(x+z)^2+\frac{\partial}{\partial x}(y+z)^2=\frac{\partial}{\partial x}8\implies 2(1+\frac{\partial z}{\partial x})(x+z)+2\frac{\partial z}{\partial x}(y+z)=0.$$

At $(1,1,1)$ the last equation becomes

$$4(1+\frac{\partial z}{\partial x})+4\frac{\partial z}{\partial x}=0\implies \frac{\partial z}{\partial x}=-\frac{1}{2}.$$

2.6.1 $\nabla f(x,y,z)=\begin{bmatrix}e^{yz}\\ xze^{yz}\\ xye^{yz}\end{bmatrix}\implies (\nabla f)(2,1,1)=\begin{bmatrix}e\\ 2e\\ 2e\end{bmatrix}.$

2.6.2 $(\nabla\times f)(x,y,z)=\begin{bmatrix}0\\ x\\ ye^{xy}\end{bmatrix}\implies (\nabla\times f)(2,1,1)=\begin{bmatrix}0\\ 2\\ e^2\end{bmatrix}.$

2.6.4 The vector $\begin{bmatrix}1\\-7\\0\end{bmatrix}$ is perpendicular to the plane. Put $f(x,y,z) = x^2+y^2-5xy+xz-yz+3$. Then $(\nabla f)(x,y,z) = \begin{bmatrix}2x-5y+z\\2y-5x-z\\x-y\end{bmatrix}$. Observe that $\nabla f(x,y,z)$ is parallel to the vector $\begin{bmatrix}1\\-7\\0\end{bmatrix}$, and hence there exists a constant a such that

$$\begin{bmatrix}2x-5y+z\\2y-5x-z\\x-y\end{bmatrix} = a\begin{bmatrix}1\\-7\\0\end{bmatrix} \implies x=a, \quad y=a, \quad z=4a.$$

Since the point is on the plane

$$x-7y=-6 \implies a-7a=-6 \implies a=1.$$

Thus $x=y=1$ and $z=4$.

2.6.7 Observe that

$$f(0,0)=1, \quad f_x(x,y) = (\cos 2y)e^{x\cos 2y} \implies f_x(0,0)=1,$$
$$f_y(x,y) = -2x\sin 2ye^{x\cos 2y} \implies f_y(0,0)=0.$$

Hence

$$f(x,y) \approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0) \implies f(x,y)\approx 1+x.$$

This gives $f(0.1,-0.2)\approx 1+0.1=1.1$.

2.6.8 This is essentially the product rule: $\mathrm{d}uv = u\mathrm{d}v + v\mathrm{d}u$, where ∇ acts the differential operator and $\times$ is the product. Recall that when we defined the volume of a parallelepiped spanned by the vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, we saw that

$$\overrightarrow{a}\bullet(\overrightarrow{b}\times\overrightarrow{c}) = (\overrightarrow{a}\times\overrightarrow{b})\bullet\overrightarrow{c}.$$

Treating $\nabla = \nabla_{\overrightarrow{u}} + \nabla_{\overrightarrow{v}}$ as a vector, first keeping $\overrightarrow{v}$ constant and then keeping $\overrightarrow{u}$ constant we then see that

$$\nabla_{\overrightarrow{u}}\bullet(\overrightarrow{u}\times\overrightarrow{v}) = (\overrightarrow{\nabla}\times\overrightarrow{u})\bullet\overrightarrow{v}, \qquad \nabla_{\overrightarrow{v}}\bullet(\overrightarrow{u}\times\overrightarrow{v}) = -\nabla\bullet(\overrightarrow{v}\times\overrightarrow{u}) = -(\overrightarrow{\nabla}\times\overrightarrow{v})\bullet\overrightarrow{u}.$$

Thus

$$\nabla\bullet(\mathrm{u}\times\mathrm{v}) = (\nabla_{\overrightarrow{u}}+\nabla_{\overrightarrow{v}})\bullet(\mathrm{u}\times\mathrm{v}) = \nabla_{\overrightarrow{u}}\bullet(\overrightarrow{u}\times\overrightarrow{v}) + \nabla_{\overrightarrow{v}}\bullet(\overrightarrow{u}\times\overrightarrow{v}) = (\overrightarrow{\nabla}\times\overrightarrow{u})\bullet\overrightarrow{v} - (\overrightarrow{\nabla}\times\overrightarrow{v})\bullet\overrightarrow{u}.$$

2.6.11 An angle of $\frac{\pi}{6}$ with the x-axis and $\frac{\pi}{3}$ with the y-axis.

2.8.7 We have

$$(\nabla f)(x,y) = \begin{bmatrix}4x^3-4(x-y)\\4y^3+4(x-y)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 4x^3 = 4(x-y) = -4y^3 \implies x=-y.$$

Hence

$$4x^3-4(x-y)=0 \implies 4x^3-8x=0 \implies 4x(x^2-2)=0 \implies x\in\{-\sqrt{2},0,\sqrt{2}\}.$$

Since $x=-y$, the critical points are thus $(-\sqrt{2},\sqrt{2})$, $(0,0)$, $(\sqrt{2},-\sqrt{2})$. The Hessian is now,

$$\mathcal{H}f(x,y) = \begin{bmatrix}12x^2-4 & 4\\ 4 & 12y^2-4\end{bmatrix},$$

and its principal minors are $\Delta_1 = 12x^2-4$ and $\Delta_2 = (12x^2-4)(12y^2-4)-16$.

If $(x,y)=(-\sqrt{2},\sqrt{2})$ or $(x,y)=(\sqrt{2},-\sqrt{2})$, then $\Delta_1 = 20>0$ and $\Delta_2 = 384>0$, so the matrix is positive definite and we have a local minimum at each of these points.

If $(x,y)=(0,0)$ then $\Delta_1=-4<0$ and $\Delta_2=0$, so the matrix is negative semidefinite and further testing is needed. What happens in a neighbourhood of $(0,0)$? We have

$$f(x,x)=2x^4>0, \qquad f(x,-x)=2x^4-4x^2=2x^2(x^2-1).$$

If x is close enough to 0, $=2x^2(x^2-1)<0$, which means that the function both increases and decreases to 0 in a neighbourhood of $(0,0)$, meaning that there is a saddle point there.

2.8.8 We have

$$\nabla f(x,y,z) = \begin{bmatrix} 8xz - 2y - 8x \\ -2x + 1 \\ 4x^2 - 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x = 1/2; \quad y = -1; \quad z = 1/2.$$

The hessian is

$$\mathscr{H} = \begin{bmatrix} 8z - 8 & -2 & 8x \\ -2 & 0 & 0 \\ 8x & 0 & -2 \end{bmatrix}.$$

The principal minors are $8z - 8$; -4, and 8. At $z = 1/2$, the matrix is negative definite and the critical point is thus a saddle point.

2.8.9 We have

$$(\nabla f)(x,y,z) = \begin{bmatrix} 2x + yz \\ 2y + xz \\ 2z + xy \end{bmatrix},$$

and

$$\mathscr{H}_r f = \begin{bmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{bmatrix}.$$

We see that $\Delta_1(x,y,z) = 2, \Delta_2(x,y,z) = \det \begin{bmatrix} 2 & z \\ z & 2 \end{bmatrix} = 4 - z^2$ and $\Delta_3(x,y,z) = \det \mathscr{H}_r f = 8 - 2x^2 - 2y^2 - 2z^2 + 2xyz$.

If $(\nabla f)(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then we must have

$$\begin{aligned} 2x &= -yz, \\ 2y &= -xz, \\ 2z &= -xy, \end{aligned}$$

and upon multiplication of the three equations,

$$8xyz = -x^2y^2z^2,$$

that is,

$$xyz(xyz + 8) = 0.$$

Clearly, if $xyz = 0$, then we must have at least one of the variables equalling 0, in which case, by virtue of the original three equations, all equal 0. Thus $(0,0,0)$ is a critical point. If $xyz = -8$, then none of the variables is 0, and solving for x, say, we must have $x = -\dfrac{8}{yz}$, and substituting this into $2x + yz = 0$ we gather $(yz)^2 = 16$, meaning that either $yz = 4$, in which case $x = -2$, or $zy = -4$, in which case $x = 2$. It is easy to see then that either exactly one of the variables is negative, or all three are negative. The other critical points are therefore $(-2,2,2)$, $(2,-2,2)$, $(2,2,-2)$, and $(-2,-2,-2)$.

At $(0,0,0)$, $\Delta_1(0,0,0) = 2 > 0, \Delta_2(0,0,0) = 4 > 0, \Delta_1(0,0,0) = 8 > 0$, and thus $(0,0,0)$ is a minimum point. If $x^2 = y^2 = z^2 = 4, xyz = -8$, then $\Delta_2(x,y,z) = 0, \Delta_3 = -32$, so these points are saddle points.

2.8.10 We have

$$(\nabla f)(x,y,z) = \begin{bmatrix} 2xy + 2 \\ x^2 + 2yz \\ y^2 - 1 \end{bmatrix},$$

and

$$\mathscr{H}_r f = \begin{bmatrix} 2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 0 \end{bmatrix}.$$

We see that $\Delta_1(x,y,z) = 2y$, $\Delta_2(x,y,z) = \det\begin{bmatrix}2y & 2x\\ 2x & 2z\end{bmatrix} = 4yz - 4x^2$ and $\Delta_3(x,y,z) = \det \mathscr{H}_r f = -8y^3$.

If $(\nabla f)(x,y,z) = \begin{bmatrix}0\\0\\0\end{bmatrix}$ then we must have

$$\begin{aligned} xy &= -1,\\ x^2 &= -2yz,\\ y &= \pm 1, \end{aligned}$$

and hence $(1,-1,\frac{1}{2})$, and $(-1,1,-\frac{1}{2})$ are the critical points. Now, $\Delta_1(1,-1,\frac{1}{2}) = -2$, $\Delta_2(1,-1,\frac{1}{2}) = -6$, and $\Delta_3(1,-1,\frac{1}{2}) = 8$. Thus $(1,-1,\frac{1}{2})$ is a saddle point. Similarly, $\Delta_1(-1,1,-\frac{1}{2}) = 2$, $\Delta_2(-1,1,-\frac{1}{2}) = -6$, and $\Delta_3(-1,1,-\frac{1}{2}) = -8$, shewing that $(1,-1,\frac{1}{2})$ is also a saddle point.

2.8.11 We find

$$\nabla f(x,y,z) = \begin{bmatrix}4yz - 4x^3\\ 4xz - 4y^3\\ 4xy - 4z^3\end{bmatrix}.$$

Assume $\nabla f(x,y,z) = 0$. Then

$$4yz = 4x^3,\ 4xz = 4y^3,\ 4xy = 4z^3 \implies xyz = x^4 = y^4 = z^4.$$

Thus $xyz \geq 0$, and if one of the variables is 0 so are the other two. Thus $(0,0,0)$ is the only critical point with at least one of the variables 0. Assume now that $xyz \neq 0$. Then

$$(xyz)^3 = x^4y^4z^4 = (xyz)^4 \implies xyz = 1 \implies yz = \frac{1}{x} \implies x^4 = 1 \implies x = \pm 1.$$

Similarly, $y = \pm 1, z = \pm 1$. Since $xyz = 1$, exactly two of the variables can be negative. Thus we find the following critical points:

$$(0,0,0), (1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1).$$

The Hessian is

$$\mathscr{H}_x f = \begin{bmatrix}-12x^2 & 4z & 4y\\ 4z & -12y^2 & 4x\\ 4y & 4x & -12z^2\end{bmatrix}.$$

If $1 = xyz = x^2 = y^2 = z^2$, we have $\Delta_1 = -12x^2 = -12 < 0$, $\Delta_2 = 144x^2y^2 - 16z^2 = 144 - 16 = 128 > 0$, and

$$\begin{aligned} \Delta_3 &= -1728x^2y^2z^2 + 192x^4 + 192z^4 + 128zyx + 192y^4\\ &= -1728 + 192 + 192 + 128 + 192\\ &= -1024\\ &< 0. \end{aligned}$$

This means that for $xyz \neq 0$ the Hessian is negative definite and the function has a local maximum at each of the four points $(1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1)$. Observe that at these critical points $f = 1$. Now $f(0,0,0) = 0$ and $f(-1,1,1) = -7$.

2.8.12 Rewrite: $f(x,y,z) = xyz(4-x-y-z) = 4xyz - x^2yz - xy^2z - xyz^2$. Then,

$$(\nabla f)(x,y,z) = \begin{bmatrix}4yz - 2xyz - y^2z - yz^2\\ 4xz - x^2z - 2xyz - xz^2\\ 4xy - x^2y - xy^2 - 2xyz\end{bmatrix},$$

$$\mathcal{H}f(x,y,z) = \begin{bmatrix}-2yz & z(4-2x-2y-z) & y(4-2x-y-2z)\\ z(4-2x-2y-z) & -2xz & x(4-x-2y-2z)\\ y(4-2x-y-2z) & x(4-x-2y-2z) & -2xy\end{bmatrix}$$

Equating the gradient to zero, we obtain,

$$yz(4-2x-y-z) = 0, \quad xz(4-x-2y-z) = 0, \quad xy(4-x-y-2z) = 0.$$

If $xyz \neq 0$ then we must have

$$4-2x-y-z=0, \quad 4-x-2y-z=0, \quad 4-x-y-2z=0 \implies x=y=z=1.$$

In this case

$$\mathcal{H}f(1,1,1) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

and the principal minors are $\Delta_1 = -2 < 0$, $\Delta_2 = 3 > 0$, and $\Delta_3 = -4 < 0$, so the matrix is negative definite and we have a local maximum at $(1,1,1)$.

If either of x, y, or z is 0, we will get $\Delta_3 = 0$, so further testing is needed. Now,

$$f(x,x,x) = x^3(4-3x), \qquad f(x,-x,x) = x^3(-4+x).$$

Thus as $x \to 0+$ then $f(x,x,x) > 0$ and $f(x,-x,x) < 0$, which means that in some neighbourhood of $(0,0,0)$ the function is both decreasing towards 0 and increasing towards 0, which means that $(0,0,0)$ is a saddle point.

2.8.13 To facilitate differentiation observe that $g(x,y,z) = (xe^{-x^2})(ye^{-y^2})(ze^{-z^2})$. Now

$$\nabla g(x,y,z) = \begin{bmatrix} (1-2x^2)(yz)(e^{-x^2})(e^{-y^2})(e^{-z^2}) \\ (1-2y^2)(xz)(e^{-x^2})(e^{-y^2})(e^{-z^2}) \\ (1-2z^2)(xy)(e^{-x^2})(e^{-y^2})(e^{-z^2}) \end{bmatrix}.$$

The function is 0 if any of the variables is 0. Since the function clearly assumes positive and negative values, we can discard any point with a 0. If $\nabla(x,y,z) = 0$, then $x = \pm\dfrac{1}{\sqrt{2}}$; $y = \pm\dfrac{1}{\sqrt{2}}$ $z = \pm\dfrac{1}{\sqrt{2}}$. We find

$$\mathscr{H}_x g = t(x,y,z) \begin{bmatrix} (4x^3-6x)(yz) & (1-2x^2)(1-2y^2)z & (1-2x^2)(1-2z^2)y \\ (1-2y^2)(1-2x^2)z & (4y^3-6y)(xz) & (1-2y^2)(1-2z^2)x \\ (1-2z^2)(1-2x^2)y & (1-2z^2)(1-2y^2)x & (4z^3-6z)(xy) \end{bmatrix},$$

with $t(x,y,z) = (e^{-x^2})(e^{-y^2})(e^{-z^2})$. Since at the critical points we have $1-2x^2 = 1-2y^2 = 1-2z^2 = 0$, the Hessian reduces to

$$\mathscr{H}_x g = (e^{-3/2}) \begin{bmatrix} (4x^3-6x)(yz) & 0 & 0 \\ 0 & (4y^3-6y)(xz) & 0 \\ 0 & 0 & (4z^3-6z)(xy) \end{bmatrix}.$$

We have

$$\Delta_1 = (4x^3-6x)(yz)$$
$$\Delta_2 = (4x^3-6x)(4y^3-6y)(xyz^2)$$
$$\Delta_3 = (4x^3-6x)(4y^3-6y)(4z^3-6z)(x^2y^2z^2).$$

Also,

$$4\left(\frac{1}{\sqrt{2}}\right)^3 - 6\left(\frac{1}{\sqrt{2}}\right) = -2\sqrt{2} < 0, \quad 4\left(-\frac{1}{\sqrt{2}}\right)^3 - 6\left(-\frac{1}{\sqrt{2}}\right) = 2\sqrt{2} > 0.$$

This means that if an even number of the variables is negative (0 or 2), then we the Hessian is negative definite, and if an odd numbers of the variables is positive (1 or 3), the Hessian is positive definite. We conclude that we have local maxima at

$$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

and local minima at

$$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

2.8.14 By the Fundamental Theorem of Calculus, there exists a continuously differentiable function G such that

$$f(x, y) = \int_{y^2-x}^{x^2+y} g(t)\mathrm{d}t = G(x^2+y) - G(y^2-x).$$

Hence

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= 2xG'(x^2+y) + G'(y^2-x) = 2xg(x^2+y) + g(y^2-x);\\ \frac{\partial f}{\partial y}(x, y) &= G'(x^2+y) - 2yG'(y^2-x) = g(x^2+y) - 2yg(y^2-x).\end{aligned}$$

This gives

$$\frac{\partial f}{\partial x}(0, 0) = g(0) = \frac{\partial f}{\partial y}(0, 0) = 0,$$

so $(0, 0)$ is a critical point. Now, the Hessian of f is

$$\mathscr{H}_f(x, y) = \begin{bmatrix} 2g(x^2+y) + 4x^2g'(x^2+y) - g'(y^2-x) & 2xg'(x^2+y) + 2yg'(y^2-x) \\ 2xg'(x^2-y) + 2yg'(y^2-x) & g'(x^2+y) - 2g(y^2-x) - 4y^2g'(y^2-x)\end{bmatrix},$$

and so

$$\mathscr{H}_f(0, 0) = \begin{bmatrix} -g'(0) & 0 \\ 0 & g'(0)\end{bmatrix}.$$

Regardless of the sign of $g'(0)$, the determinant of of this last matrix is $-(g'(0))^2 < 0$, and so $(0, 0)$ is a saddle point.

2.8.15 Since the coordinates $(x, \dfrac{\sqrt{144-16x^2}}{3})$, $-3 \le x \le 3$ describe an ellipse centred at the origin and semi-axes 3 and 4, and the coordinates $(y, \sqrt{4-y^2})$, $-2 \le y \le 2$ describe a circle centred at the origin with radius 2, the problem reduces to finding the minimum between the boundaries of the circle and the ellipse. Geometrically this is easily seen to be 1.

2.9.1 We have

$$\begin{aligned}\nabla(abc) = \lambda\nabla(2ab + 2bc + 2ca - S) \quad &\implies \begin{bmatrix} bc \\ ca \\ ab\end{bmatrix} = \lambda \begin{bmatrix} 2b + 2c \\ 2a + 2c \\ 2b + 2a\end{bmatrix}\\ &\implies \begin{array}{rcl} bc &=& 2\lambda(b+c) \\ ca &=& 2\lambda(a+c) \\ ab &=& 2\lambda(b+a)\end{array}\end{aligned}$$

By physical considerations, $abc \neq 0$ and so $\lambda \neq 0$. Hence, by successively dividing the equations,

$$\frac{b}{a} = \frac{b+c}{c+a} \implies a = b, \quad \frac{c}{b} = \frac{a+c}{b+a} \implies b = c, \quad \frac{a}{c} = \frac{b+a}{b+c} \implies a = c.$$

Therefore

$$2a^2 + 2a^2 + 2a^2 = S \implies a = \frac{\sqrt{S}}{\sqrt{6}},$$

and the maximum volume is

$$abc = \frac{(\sqrt{S})^3}{(\sqrt{6})^3}.$$

The above result can be simply obtained by using the AM-GM inequality:

$$\frac{S}{3} = \frac{2ab + 2bc + 2ca}{3} \ge ((2ab)(2bc)(2ca))^{1/3} = 2(abc)^{2/3} \implies abc \le \frac{S^{3/2}}{6^{3/2}}.$$

Equality happens if

$$2ab = 2bc = 2ca \implies a = b = c = \frac{\sqrt{S}}{\sqrt{6}}.$$

2.9.2

1. The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is perpendicular to $\mathbf{\Pi}$. Hence, the equation of the perpendicular passing through P is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies x = 1 + ta, \quad y = 1 + tb, \quad z = 1 + tc.$$

The intersection of the line and the plane happens when

$$a(1 + at) + b(1 + tb) + c(1 + tc) = d \implies t = \frac{d - a - b - c}{a^2 + b^2 + c^2}.$$

Hence

$$P' = \begin{pmatrix} 1 + a \cdot \dfrac{d - a - b - c}{a^2 + b^2 + c^2} \\ 1 + b \cdot \dfrac{d - a - b - c}{a^2 + b^2 + c^2} \\ 1 + c \cdot \dfrac{d - a - b - c}{a^2 + b^2 + c^2} \end{pmatrix}$$

The distance is then

$$\sqrt{\left(a \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}\right)^2 + \left(b \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}\right)^2 + \left(c \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}\right)^2}$$

2. Let $f(x, y, z) = (x - 1)^2 + (y - 1)^2 + (z - 1)^2$ be the square of the distance from P to a point on the plane and let $g(x, y, z) = ax + by + cz - d$. Using Lagrange multipliers,

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \begin{bmatrix} 2(x - 1) \\ 2(y - 1) \\ 2(z - 1) \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies 2(x-1) = \lambda a, \quad 2(y-1) = \lambda b, \quad 2(z-1) = \lambda c.$$

Since $(1, 1, 1)$ is not on the plane and $abc \neq 0$, we gather that $\lambda \neq 0$. Now,

$$x = 1 + \frac{\lambda a}{2}, \quad y = 1 + \frac{\lambda b}{2}, \quad z = 1 + \frac{\lambda c}{2}.$$

Putting these into the equation of the plane,

$$a\left(1 + \frac{\lambda a}{2}\right) + b\left(1 + \frac{\lambda b}{2}\right) + c\left(1 + \frac{\lambda c}{2}\right) = d \implies \lambda = 2 \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}.$$

Then the coordinates of P' are

$$x = 1 + \frac{\lambda a}{2} = 1 + a \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}, \quad y = 1 + \frac{\lambda b}{2} = 1 + b \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}, \quad z = 1 + \frac{\lambda c}{2} = 1 + c \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2},$$

as before.

3. Consider the function

$$t(x, y) = (x - 1)^2 + (y - 1)^2 + \left(\frac{d - ax - by}{c} - 1\right)^2,$$

which is the square of the distance from a point (x, y, z) on the plane to the point $(1, 1, 1)$.
Now,

$$\nabla t(x, y) = \begin{bmatrix} 2(x - 1) - 2\dfrac{a}{c}\left(\dfrac{d - ax - by}{c} - 1\right) \\ 2(y - 1) - 2\dfrac{b}{c}\left(\dfrac{d - ax - by}{c} - 1\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies

$$x = \frac{-b^2 - c^2 + ab - ad + ac}{a^2 + b^2 + c^2} = 1 + a \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}, \qquad y = \frac{c^2 + a^2 - ab + bd - bc}{a^2 + b^2 + c^2} = 1 + b \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2},$$

as before. Substituting this in the equation of the plane gives the same coordinate of z, as before.

2.9.3 Using CBS,

$$\frac{x+3y}{2} \leq (\frac{x^4+81y^4}{2})^{1/4} = \frac{36^{1/4}}{2^{1/4}} \implies x+3y \leq 2^{3/4}\sqrt{6} = 2^{5/4}\sqrt{3}.$$

2.9.4 Using AM-GM,

$$\frac{1}{6^{1/3}} = \sqrt[6]{x^2y^3z} \leq \frac{2x+3y+z}{6} \implies 2x+3y+z \geq 6^{2/3}.$$

2.9.5 We put $g(x,y) = 5x^2+6xy+5y^2-8$ and argue using Lagrange multipliers. We have

$$\nabla f(x,y) = \lambda\nabla g(x,y) \implies \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 10x+6y \\ 6x+10y \end{bmatrix}.$$

This gives the three equations

$$0 = 5(\lambda-1)x+3y; \quad 0 = 3x+5(\lambda-1)y; \quad 5x^2+6xy+5y^2 = 8.$$

The linear system (the first two equations) will have the unique solution $(0,0)$ as long as $25(\lambda-1)^2-9 \neq 0$, but this solution does not lie on the third equation. If $25(\lambda-1)^2-9=0$, then we deduce that $x = \pm y$. Substituting this into the third equation we gather that $10x^2 \pm 6x^2 = 8$, resulting in $x = \pm\sqrt{2}$ or $x = \pm\frac{1}{\sqrt{2}}$. Taking into account the third equation, the feasible values are $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$ The desired maximum is thus

$$f(-\sqrt{2}, \sqrt{2}) = f(\sqrt{2}, -\sqrt{2}) = 4$$

and the minimum is

$$f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = 1.$$

Aliter: Observe that, using AM-GM,

$$5x^2+6xy+5y^2 = 8 \implies x^2+y^2 = \frac{8}{5} - \frac{6}{5}xy \geq \frac{8}{5} - \frac{6}{5}\cdot\frac{x^2+y^2}{2} \implies x^2+y^2 \geq \frac{5}{8}\cdot\frac{8}{5} = 1.$$

2.9.6 Put $g(x,y) = x^p+y^p-1$. We need $a = p\lambda x^{p-1}$ and $b = p\lambda y^{p-1}$. Clearly then , $\lambda \neq 0$. We then have

$$x = \left(\frac{a}{\lambda p}\right)^{1/(p-1)}, \quad y = \left(\frac{b}{\lambda p}\right)^{1/(p-1)}.$$

Thus

$$1 = x^p+y^p = \left(\frac{a}{\lambda p}\right)^{p/(p-1)} + \left(\frac{b}{\lambda p}\right)^{p/(p-1)},$$

which gives

$$\lambda = \left(\left(\frac{a}{p}\right)^{p/(p-1)} + \left(\frac{b}{p}\right)^{p/(p-1)}\right)^{(p-1)/p}.$$

This gives

$$x = \frac{a^{1/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}}, \quad y = \frac{b^{1/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}}.$$

Since f is non-negative, these points define a maximum for f and so

$$ax+by \leq \frac{a^{p/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}} + \frac{b^{p/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}}.$$

2.9.7 Let $g(x,y,z) = (x-1)^2+(y-2)^2+(z-3)^2-4$. We solve

$$\nabla f \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda\nabla g \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for x, y, λ. This requires

$$\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \begin{bmatrix} 2(x-1)\lambda \\ 2(y-2)\lambda \\ 2(z-3)\lambda \end{bmatrix}.$$

Clearly, $\lambda \neq 1$. This gives $x = \frac{-\lambda}{1-\lambda}$, $y = \frac{-2\lambda}{1-\lambda}$, and $z = \frac{-3\lambda}{1-\lambda}$. Substituting into $(x-1)^2+(y-2)^2+(z-3)^2 = 4$, we gather that

$$\left(\frac{-\lambda}{1-\lambda} - 1\right)^2 + \left(\frac{-2\lambda}{1-\lambda} - 2\right)^2 + \left(\frac{-3\lambda}{1-\lambda} - 3\right)^2 = 4,$$

from where

$$\lambda = 1 \pm \frac{\sqrt{14}}{2}.$$

This gives the two points

$$(x, y, z) = \left(1 + \frac{2}{\sqrt{14}}, 2 + \frac{4}{\sqrt{14}}, 3 + \frac{6}{\sqrt{14}}\right)$$

and

$$(x, y, z) = \left(1 - \frac{2}{\sqrt{14}}, 2 - \frac{4}{\sqrt{14}}, 3 - \frac{6}{\sqrt{14}}\right).$$

The first point gives an absolute maximum of $18 + \frac{12\sqrt{14}}{7}$ and the second an absolute minimum of $18 - \frac{12\sqrt{14}}{7}$.

2.9.8 Observe that the ellipse is symmetric about the origin. Now maximise and minimise the distance between a point on the ellipse and the origin. If a and b are the semi-axes, you will find that $2a = 2$ and $2b = 6$

2.9.9 Put $g(x, y, z) = x^2 + y^2 - 2, h(x, y, z) = x + z - 1$. We must find λ, δ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \delta \nabla h(x, y, z),$$

which translates into

$$\begin{aligned} 1 &= 2\lambda x + \delta, \\ 1 &= 2\lambda y, \\ 1 &= \delta, \end{aligned}$$

and

$$\begin{aligned} x^2 + y^2 &= 1, \\ x + z &= 1. \end{aligned}$$

We deduce that $x = 0, y = \pm\sqrt{2}, z = 1$. We may shew that $(0, \sqrt{2}, 1)$ yields a maximum and that $(0, -\sqrt{2}, 1)$ yields a minimum.

2.9.10 One can use Lagrange multipliers here. But perhaps the easiest approach is to put $y = 1-x$ and maximise

$$f(x) = x + \sqrt{x(1-x)}.$$

For this we have

$$f'(x) = 0 \implies 1 + \frac{1-2x}{2\sqrt{x(1-x)}} = 0 \implies x = \frac{1}{2} + \frac{\sqrt{2}}{4}.$$

Since

$$f''(x) = -\frac{(1-2x)^2}{4(x(1-x))^{3/2}} - \frac{1}{\sqrt{x(1-x)}} < 0,$$

the value sought is a maximum. This maximum is thus

$$f\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) = \frac{1}{2} + \frac{\sqrt{2}}{2}.$$

2.9.11 Claim: the function achieves its maximum on the boundary of the triangle. To prove this claim we have to prove that there are no critical points strictly inside the triangle. For this we compute the gradient and set it equal to the zero vector:

$$(\nabla f)(x, y) = \begin{bmatrix} -ax^{a-1}y^b e^{-(x+y)} \\ -bx^a y^{b-1} e^{-(x+y)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = 0 \text{ or } y = 0,$$

which means that the critical points occur on the boundary. Since the function is identically 0 for $x = 0$ or $y = 0$, we only need to look on the line $x + y = 1$ for the maxima. Hence we maximise f subject to the constraint $x + y = 1$. Since $x + y = 1$, we can see that $f(x, y) = x^a y^b e^{-(x+y)} = x^a y^b e^{-1}$ on the line, so the problem reduces to maximising $h(x, y) = x^a y^b$ subject to the constraint $x + y = 1$. Using Lagrange multipliers,

$$(\nabla h)(x, y) = \lambda(\nabla g)(x, y) \implies \begin{bmatrix} ax^{a-1}y^b \\ bx^a y^{b-1} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which in turn

$$\implies ax^{a-1}y^b = \lambda = bx^a y^{b-1} \implies ay = bx \implies ay = b(1-y) \implies y = \frac{b}{a+b}, \quad x = \frac{a}{a+b}.$$

Finally,

$$f(x, y) = x^a y^b e^{-(x+y)} \leq x^a y^b e^{-1} \leq \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b e^{-1}.$$

2.9.15 Try $p(x, y) = (y^2 + 1)x^2 + 2xy + 1$.

3.3.1

1. Let $L_1 : y = x + 1$, $L_2 : -x + 1$. Then

$$\begin{aligned} \int_C x\mathrm{d}x + y\mathrm{d}y &= \int_{L_1} x\mathrm{d}x + y\mathrm{d}y + \int_{L_2} x\mathrm{d}x + y\mathrm{d}y \\ &= \int_{-1}^{1} x\mathrm{d}x(x+1)\mathrm{d}x + \int_0^1 x\mathrm{d}x - (-x+1)\mathrm{d}x \\ &= 0. \end{aligned}$$

Also, both on L_1 and on L_2 we have $||\mathrm{dx}|| = \sqrt{2}\mathrm{d}x$, thus

$$\begin{aligned} \int_C xy||\mathrm{dx}|| &= \int_{L_1} xy||\mathrm{dx}|| + \int_{L_2} xy||\mathrm{dx}|| \\ &= \sqrt{2}\int_{-1}^{1} x(x+1)\mathrm{d}x - \sqrt{2}\int_0^1 x(-x+1)\mathrm{d}x \\ &= 0. \end{aligned}$$

2. We put $x = \sin t$, $y = \cos t$, $t \in \left[-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right]$. Then

$$\begin{aligned} \int_C x\mathrm{d}x + y\mathrm{d}y &= \int_{-\pi/2}^{\pi/2} (\sin t)(\cos t)\mathrm{d}t - (\cos t)(\sin t)\mathrm{d}t \\ &= 0. \end{aligned}$$

Also, $||\mathrm{dx}|| = \sqrt{(\cos t)^2 + (-\sin t)^2}\mathrm{d}t = \mathrm{d}t$, and thus

$$\begin{aligned} \int_C xy||\mathrm{dx}|| &= \int_{-\pi/2}^{\pi/2} (\sin t)(\cos t)\mathrm{d}t \\ &= \frac{(\sin t)^2}{2}\Big|_{-\pi/2}^{\pi/2} \\ &= 0. \end{aligned}$$

3.3.2 Let Γ_1 denote the straight line segment path from O to $A = (2\sqrt{3}, 2)$ and Γ_2 denote the arc of the circle centred at $(0, 0)$ and radius 4 going counterclockwise from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{5}$.

Observe that the Cartesian equation of the line $\overleftrightarrow{\mathrm{OA}}$ is $y = \frac{x}{\sqrt{3}}$. Then on Γ_1

$$x\mathrm{d}x + y\mathrm{d}y = x\mathrm{d}x + \frac{x}{\sqrt{3}}\mathrm{d}\frac{x}{\sqrt{3}} = \frac{4}{3}x\mathrm{d}x.$$

Hence

$$\int_{\Gamma_1} x\mathrm{d}x + y\mathrm{d}y = \int_0^{2\sqrt{3}} \frac{4}{3}x\mathrm{d}x = 8.$$

On the arc of the circle we may put $x = 4\cos\theta$, $y = 4\sin\theta$ and integrate from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{5}$. Observe that there

$$x\mathrm{d}x + y\mathrm{d}y = (\cos\theta)\mathrm{d}\cos\theta + (\sin\theta)\mathrm{d}\sin\theta = -\sin\theta\cos\theta\mathrm{d}\theta + \sin\theta\cos\theta\mathrm{d}\theta = 0,$$

and since the integrand is 0, the integral will be zero.

Assembling these two pieces,

$$\int_{\Gamma} x\mathrm{d}x + y\mathrm{d}y = \int_{\Gamma_1} x\mathrm{d}x + y\mathrm{d}y + \int_{\Gamma_2} x\mathrm{d}x + y\mathrm{d}y = 8 + 0 = 8.$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <x,y> ), Line( <0,0>, <2*sqrt(3),2> ))
> +LineInt( VectorField( <x,y> ), Arc(Circle( <0,0>, 4), Pi/6, Pi/5) );
```

3.3.3 Using the parametrisations from the solution of problem 3.3.3, we find on Γ_1 that

$$x||\mathrm{d}\mathbf{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = x\sqrt{1 + \frac{1}{3}}\mathrm{d}x = \frac{2}{\sqrt{3}}x\mathrm{d}x,$$

whence

$$\int_{\Gamma_1} x||\mathrm{d}\mathbf{x}|| = \int_0^{2\sqrt{3}} \frac{2}{\sqrt{3}}x\mathrm{d}x = 4\sqrt{3}.$$

On Γ_2 that

$$x||\mathrm{d}\mathbf{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = 16\cos\theta\sqrt{\sin^2\theta + \cos^2\theta}\mathrm{d}\theta = 16\cos\theta\mathrm{d}\theta,$$

whence

$$\int_{\Gamma_2} x||\mathrm{d}\mathbf{x}|| = \int_{\pi/6}^{\pi/5} 16\cos\theta\mathrm{d}\theta = 16\sin\frac{\pi}{5} - 16\sin\frac{\pi}{6} = 4\sin\frac{\pi}{5} - 8.$$

Assembling these we gather that

$$\int_{\Gamma} x||\mathrm{d}\mathbf{x}|| = \int_{\Gamma_1} x||\mathrm{d}\mathbf{x}|| + \int_{\Gamma_2} x||\mathrm{d}\mathbf{x}|| = 4\sqrt{3} - 8 + 16\sin\frac{\pi}{5}.$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> PathInt(x, [x,y]= Line( <0,0>, <2*sqrt(3),2> ))
> +PathInt(x, [x,y]=Arc(Circle( <0,0>, 4), Pi/6, Pi/5) );
```

Maple gives $16\cos\frac{3\pi}{10}$ rather than our $16\sin\frac{\pi}{5}$. To check that these two are indeed the same, use the code

```
> is(16*cos(3*Pi/10)=16*sin(Pi/5));
```

which returns `true`.

3.3.4 The curve lies on the sphere, and to parametrise this curve, we dispose of one of the variables, y say, from where $y = 1 - x$ and $x^2 + y^2 + z^2 = 1$ give

$$\begin{aligned} x^2 + (1-x)^2 + z^2 = 1 \quad &\implies 2x^2 - 2x + z^2 = 0 \\ &\implies 2\left(x - \frac{1}{2}\right)^2 + z^2 = \frac{1}{2} \\ &\implies 4\left(x - \frac{1}{2}\right)^2 + 2z^2 = 1. \end{aligned}$$

So we now put

$$x = \frac{1}{2} + \frac{\cos t}{2}, \quad z = \frac{\sin t}{\sqrt{2}}, \quad y = 1 - x = \frac{1}{2} - \frac{\cos t}{2}.$$

We must integrate on the side of the plane that can be viewed from the point $(1, 1, 0)$ (observe that the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is normal to the plane). On the zx-plane, $4\left(x - \frac{1}{2}\right)^2 + 2z^2 = 1$ is an ellipse. To obtain a positive parametrisation we must integrate from $t = 2\pi$ to $t = 0$ (this is because when you look at the ellipse from the point $(1, 1, 0)$ the positive x-axis is to your left, and not your right). Thus

$$\begin{aligned}
\oint_\Gamma z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z &= \int_{2\pi}^{0} \frac{\sin t}{\sqrt{2}}\mathrm{d}\left(\frac{1}{2} + \frac{\cos t}{2}\right) \\
&\quad + \int_{2\pi}^{0} \left(\frac{1}{2} + \frac{\cos t}{2}\right)\mathrm{d}\left(\frac{1}{2} - \frac{\cos t}{2}\right) \\
&\quad + \int_{2\pi}^{0} \left(\frac{1}{2} - \frac{\cos t}{2}\right)\mathrm{d}\left(\frac{\sin t}{\sqrt{2}}\right) \\
&= \int_{2\pi}^{0} \left(\frac{\sin t}{4} + \frac{\cos t}{2\sqrt{2}} + \frac{\cos t \sin t}{4} - \frac{1}{2\sqrt{2}}\right)\mathrm{d}t \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

3.5.1 2

3.5.2 $\frac{1}{3}$

3.5.3 $\frac{15\pi}{16}$

3.5.4 The integral equals

$$\begin{aligned}
\int_D xy\mathrm{d}x\mathrm{d}y &= \int_0^1 x\left(\int_{x^2}^{\sqrt{x}} y\ \mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^1 \frac{1}{2}x(x - x^4)\ \mathrm{d}x \\
&= \frac{1}{12}.
\end{aligned}$$

3.5.5 The integral equals

$$\begin{aligned}
\int_D x\sin x\sin y\mathrm{d}x\mathrm{d}y + \int_D y\sin x\sin y\mathrm{d}x\mathrm{d}y &= 2\left(\int_0^\pi y\sin y\ \mathrm{d}y\right)\left(\int_0^\pi \sin x\ \mathrm{d}x\right) \\
&= 4\pi.
\end{aligned}$$

3.5.6 The integral is

$$\begin{aligned}
\int_{x\leq y} x^2\mathrm{d}x\mathrm{d}y + \int_{y\leq x} y^2\mathrm{d}x\mathrm{d}y &= \int_0^1\int_0^y x^2\mathrm{d}x\mathrm{d}y + \int_0^1\int_y^1 y^2\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \frac{y^3}{3}\mathrm{d}y + \int_0^1 (y^2 - y^3)\,\mathrm{d}y \\
&= \left.\frac{y^4}{12}\right|_0^1 + \left.\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\right|_0^1 \\
&= \frac{1}{12} + \frac{1}{3} - \frac{1}{4} \\
&= \frac{1}{6}.
\end{aligned}$$

3.5.7 $\frac{21}{8}$

3.5.8 Observe that

$$x^2 + y^2 = 16, y = -\frac{\sqrt{3}}{3}x + 4 \implies 16 - x^2 = \left(-\frac{\sqrt{3}}{3}x + 4\right)^2 \implies x = 0, 2\sqrt{3}.$$

The integral is

$$\begin{aligned}
\int_0^{2\sqrt{3}}\int_{-\frac{\sqrt{3}}{3}x+4}^{\sqrt{16-x^2}} x\mathrm{d}y\mathrm{d}x &= \int_0^{2\sqrt{3}} x\left(\sqrt{16 - x^2} + \frac{\sqrt{3}}{3}x - 4\right)\mathrm{d}x \\
&= \left.-\frac{1}{3}(16 - x^2)^{3/2} + \frac{\sqrt{3}}{9}x^3 - 2x^2\right|_0^{2\sqrt{3}} \\
&= \frac{8}{3}.
\end{aligned}$$

3.5.9 $e-1$

3.5.10 We have

$$\begin{aligned}
\int_{[0;1]^2} \min(x,y^2)\mathrm{d}A &= \int_{\substack{[0;1]^2\\ x\leq y^2}} x\mathrm{d}A + \int_{\substack{[0;1]^2\\ y^2<x}} y^2\mathrm{d}A \\
&= \int_0^1\int_0^{y^2} x\mathrm{d}x\mathrm{d}y + \int_0^1\int_{y^2}^1 y^2\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 x^2\Big|_0^{y^2}\mathrm{d}y + \int_0^1 y^2x\Big|_{y^2}^1\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 y^4\mathrm{d}y + \int_0^1 (y^2-y^4)\mathrm{d}y \\
&= \frac{1}{10}+\frac{2}{15} \\
&= \frac{7}{30}.
\end{aligned}$$

3.5.11 Begin by finding the Cartesian equations of the various lines: for $\overleftrightarrow{OA}$ is $y=\dfrac{x}{3}$ $(0\leq x\leq 1)$, for $\overleftrightarrow{AB}$ is $y=3x-8$ $(3\leq x\leq 4)$, and for $\overleftrightarrow{BO}$ is $y=x$ $(0\leq x\leq 4)$.

We have a choice of whether integrating with respect to x or y first. Upon examining the region, one notices that it does not make much of a difference. I will integrate with respect to y first. In such a case notice that for $0\leq x\leq 3$, y goes from the line $\overleftrightarrow{OA}$ to the line $\overleftrightarrow{OB}$, and for $3\leq x\leq 4$, y goes from the line $\overleftrightarrow{AB}$ to the line $\overleftrightarrow{OB}$

$$\begin{aligned}
\int_{\mathcal{R}} xy\mathrm{d}A &= \int_0^3\int_{x/3}^x xy\mathrm{d}y\mathrm{d}x + \int_3^4\int_{3x-8}^x xy\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2}\int_0^3 xy^2\Big|_{x/3}^x\mathrm{d}x + \frac{1}{2}\int_3^4 xy^2\Big|_{3x-8}^x\mathrm{d}x \\
&= \frac{1}{2}\int_0^3 x\left(x^2-\frac{x^2}{9}\right)\mathrm{d}x + \frac{1}{2}\int_3^4 x\left(x^2-(3x-8)^2\right)\mathrm{d}x \\
&= \frac{4}{9}\int_0^3 x^3\mathrm{d}x + \frac{1}{2}\int_3^4\left(-8x^3+48x^2-64x\right)\mathrm{d}x \\
&= 9+9 \\
&= 18.
\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
>  with(Student[VectorCalculus]):
>  int(x*y, [x,y]=Triangle(<0,0>,<3,1>,<4,4>));
```

Maple can also provide the limits of integration, but this command is limited, since Maple is quite whimsical about which order of integration to choose. It also evaluates expressions that it deems below its dignity to return unevaluated.

```
>  int(x*y, [x,y]=Triangle(<0,0>,<3,1>,<4,4>), 'inert');
```

3.5.12 Integrating by parts,

$$\begin{aligned}
\int_D \log_e(1+x+y)\mathrm{d}x\mathrm{d}y &= \int_0^1\left(\int_0^{1-x}\log_e(1+x+y)\,\mathrm{d}y\right)\,\mathrm{d}x \\
&= \int_0^1 \left[(1+x+y)\log_e(1+x+y)-(1+x+y)\right]_0^{1-x}\,\mathrm{d}x \\
&= \int_0^1 \left(2\log_e(2)-1-\log_e(1+x)-x\log_e(1+x)+x\right)\,\mathrm{d}x \\
&= \frac{1}{4}.
\end{aligned}$$

3.5.13 First observe that on $[0;2]^2$, $0 \leq \llbracket x+y^2 \rrbracket \leq 6$, so we decompose the region of integration according to where $\llbracket x+y^2 \rrbracket$ jumps across integer values. We have

$$\begin{aligned}\int_{[0;2]^2} \llbracket x+y^2 \rrbracket \mathrm{d}A &= \int\limits_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 1}} 1\mathrm{d}A + \int\limits_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 2}} 2\mathrm{d}A + \int\limits_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 3}} 3\mathrm{d}A + \int\limits_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 4}} 4\mathrm{d}A + \int\limits_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 5}} 5\mathrm{d}A \\ &= \int\limits_{\substack{[0;2]^2 \\ 1\leq x+y^2<2}} \mathrm{d}A + 2\int\limits_{\substack{[0;2]^2 \\ 2\leq x+y^2<3}} \mathrm{d}A + 3\int\limits_{\substack{[0;2]^2 \\ 3\leq x+y^2<4}} \mathrm{d}A + 4\int\limits_{\substack{[0;2]^2 \\ 4\leq x+y^2<5}} \mathrm{d}A + 5\int\limits_{\substack{[0;2]^2 \\ 5\leq x+y^2<6}} \mathrm{d}A\end{aligned}$$

By looking at the regions (as in figures A.13 through A.17 below) (I am omitting the details of the integrations, relying on Maple for the evaluations), we obtain

$$\int\limits_{\substack{[0;2]^2 \\ 1\leq x+y^2<2}} \mathrm{d}A = \int_0^1 \int_{\sqrt{1-x}}^{\sqrt{2-x}} \mathrm{d}y\mathrm{d}x + \int_1^2 \int_0^{\sqrt{2-x}} \mathrm{d}y\mathrm{d}x = -\frac{4}{3} + \frac{4}{3}\sqrt{2} + \frac{2}{3} = -\frac{2}{3} + \frac{4}{3}\sqrt{2}.$$

$$2\int\limits_{\substack{[0;2]^2 \\ 2\leq x+y^2<3}} \mathrm{d}A = 2\int_0^2 \int_{\sqrt{2-x}}^{\sqrt{3-x}} \mathrm{d}y\ \mathrm{d}x = 4\sqrt{3} - \frac{8}{3}\sqrt{2} - \frac{4}{3}.$$

$$3\int\limits_{\substack{[0;2]^2 \\ 3\leq x+y^2<4}} \mathrm{d}A = 3\int_0^2 \int_{\sqrt{3-x}}^{\sqrt{4-x}} \mathrm{d}y\ \mathrm{d}x = 18 - 6\sqrt{3} - 4\sqrt{2}.$$

$$4\int\limits_{\substack{[0;2]^2 \\ 4\leq x+y^2<5}} \mathrm{d}A = 4\int_0^1 \int_{\sqrt{4-x}}^{2} \mathrm{d}y\ \mathrm{d}x + 4\int_1^2 \int_{\sqrt{4-x}}^{\sqrt{5-x}} \mathrm{d}y\ \mathrm{d}x = -\frac{40}{3} + 8\sqrt{3} + \frac{64}{3} - 16\sqrt{3} + \frac{16}{3}\sqrt{2}.$$

$$5\int\limits_{\substack{[0;2]^2 \\ 4\leq x+y^2<5}} \mathrm{d}A = 5\int_1^2 \int_{\sqrt{5-x}}^{5} \mathrm{d}y\ \mathrm{d}x = -\frac{50}{3} + 10\sqrt{3}.$$

Adding all the above, we obtain

$$\int_{[0;2]^2} \llbracket x+y^2 \rrbracket \mathrm{d}A = \frac{22}{3} + \frac{4}{3}\sqrt{3} - \frac{4}{3}\sqrt{2} \approx 7.7571.$$

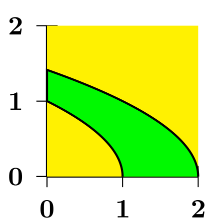

Figure A.13: $1 \leq x + y^2 < 2$.

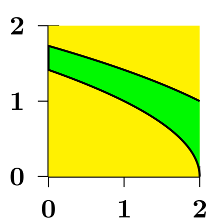

Figure A.14: $2 \leq x + y^2 < 3$.

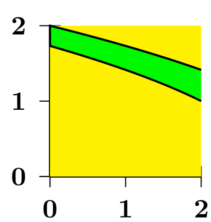

Figure A.15: $3 \leq x + y^2 < 4$.

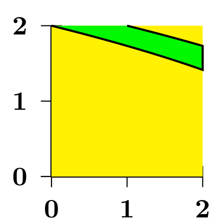

Figure A.16: $4 \leq x + y^2 < 5$.

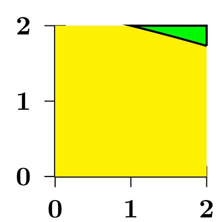

Figure A.17: $5 \leq x + y^2 < 6$.

3.5.14 Observe that in the rectangle $[0\,;1] \times [0\,;2]$ we have $0 \leq x+y \leq 3$. Hence

$$\begin{aligned}\int_R \llbracket x+y \rrbracket \mathrm{d}A &= \int_{\substack{R \\ 1\leq x+y<2}} 1\mathrm{d}A + \int_{\substack{R \\ 2\leq x+y<3}} 2\mathrm{d}A \\ &= \int_1^2 \int_{1-x}^{2-x} 1\mathrm{d}y\mathrm{d}x + \int_1^2 \int_{2-x}^{2} 2\mathrm{d}y\mathrm{d}x \\ &= 4.\end{aligned}$$

3.5.15 $\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\mathrm{d}y\mathrm{d}x + \int_1^2\int_0^{\sqrt{4-x^2}} x\mathrm{d}y\mathrm{d}x = \frac{7}{3}$.

3.5.16 $\int_0^1\int_{-2}^{3} x\mathrm{d}x\mathrm{d}y + \int_1^2\int_{-2}^{-1} x\mathrm{d}x\mathrm{d}y + \int_1^2\int_0^{1} x\mathrm{d}x\mathrm{d}y + \int_1^2\int_2^{3} x\mathrm{d}x\mathrm{d}y = 7.$

3.5.17 Exchanging the order of integration,

$$\int_0^{\pi/2}\int_0^y \frac{\cos y}{y}\mathrm{d}x\mathrm{d}y = \int_0^{\pi/2}\cos y\mathrm{d}y = 1.$$

3.5.18 Upon splitting the domain of integration, we find that the integral equals

$$\begin{aligned}\int_1^2\left(\int_y^{y^2}\sin\frac{\pi x}{2y}\mathrm{d}x\right)\mathrm{d}y &= \int_1^2\left[-\frac{2y}{\pi}\cos\frac{\pi x}{2y}\right]_y^{y^2}\mathrm{d}y \\ &= -\int_1^2 -\frac{2y}{\pi}\cos\frac{\pi y}{2}\mathrm{d}y \\ &= \frac{4(\pi+2)}{\pi^3},\end{aligned}$$

upon integrating by parts.

3.5.19 The integral is 0. Observe that if $(x, y) \in D$ then $(-x, y) \in D$. Also, $f(-x, y) = -f(x, y)$.

3.5.20 $\int_{-2/\sqrt{5}}^{2/\sqrt{5}}\int_{-\frac{1}{2}\sqrt{4-y^2}}^{\frac{1}{2}\sqrt{4-y^2}} \mathrm{d}x\mathrm{d}y = \frac{8}{5} + 4\arcsin\left(\frac{\sqrt{5}}{5}\right).$

3.5.21 The integral equals

$$\begin{aligned}\int_D xy\mathrm{d}A &= \int_0^1\left(\int_0^{\frac{1-x}{1+x}} xy\ \mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1\left(\frac{1}{2}x\left(\frac{1-x}{1+x}\right)^2\right)\mathrm{d}x \\ &= \int_1^2\frac{(t-1)(t-2)^2}{t^2}\mathrm{d}t \\ &= 4\log_e 2 - \frac{11}{4}.\end{aligned}$$

3.5.22 Using integration by parts,

$$\begin{aligned}\int_D \log_e(1+x^2+y)\mathrm{d}A &= \int_0^1\left(\int_0^{1-x^2}\log_e(1+x^2+y)\ \mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1(2\log_e(2) - 1 - \log_e(1+x^2))\mathrm{d}x \\ &\quad + \int_0^1(-x^2\log_e(1+x^2)+x^2)\ \mathrm{d}x \\ &= \frac{2}{3}\log_e 2 + \frac{8}{9} - \frac{\pi}{3}.\end{aligned}$$

3.5.23 $\int_0^2\int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}} x\mathrm{d}x\mathrm{d}y = 2.$

3.5.25 Let

$$D_1 = \{(x, y) \in \mathbb{R}^2 | -1 \le x \le 1, x \le y\},$$
$$D_2 = \{(x, y) \in \mathbb{R}^2 | -1 \le x \le 1, x > y\}.$$

Then $D = D_1 \cup D_2, D_1 \cap D_2 = \emptyset$ and so

$$\int_D f(x, y)\mathrm{d}x\mathrm{d}y = \int_{D_1} f(x, y)\mathrm{d}x\mathrm{d}y + \int_{D_2} f(x, y)\mathrm{d}x\mathrm{d}y.$$

By symmetry,
$$\int_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{D_2} f(x,y)\mathrm{d}x\mathrm{d}y,$$
and so
$$\begin{aligned}\int_D f(x,y)\mathrm{d}x\mathrm{d}y &= 2\int_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= 2\int_{-1}^{1}\left(\int_x^1 (y-x)\,\mathrm{d}y\right)\,\mathrm{d}x \\ &= \int_{-1}^{1}(1-2x+x^2)\,\mathrm{d}x \\ &= \frac{8}{3}.\end{aligned}$$

3.5.26 The line joining A, and B has equation $y=-x-2$, line joining B, and C has equation $y=-7x+10$, and line joining A, and C has equation $y=2x+1$. We split the triangle along the vertical line $x=1$, and integrate first with respect to y. The desired integral is then
$$\begin{aligned}\int_D (2x+3y+1)\,\mathrm{d}x\mathrm{d}y &= \int_{-1}^{1}\left(\int_{-x-2}^{2x+1}(2x+3y+1)\mathrm{d}y\right)\mathrm{d}x \\ &\quad + \int_1^2\left(\int_{-x-2}^{-7x+10}(2x+3y+1)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-1}^{1}\left(\frac{21}{2}x^2+9x-\frac{3}{2}\right)\mathrm{d}x \\ &\quad + \int_1^2\left(60x^2-198x+156\right)\mathrm{d}x \\ &= 4-1 \\ &= 3.\end{aligned}$$

3.5.27 Since f is positive and decreasing,
$$\int_0^1\int_0^1 f(x)f(y)(y-x)(f(x)-f(y))\mathrm{d}x\mathrm{d}y \geq 0,$$
from where the desired inequality follows.

3.5.28 The domain of integration is a triangle. The integral equals
$$\begin{aligned}\int_D xy(x+y)\mathrm{d}x\mathrm{d}y &= \int_0^1\left(\int_0^{1-x} xy(x+y)\,\mathrm{d}y\right)\,\mathrm{d}x \\ &= \int_0^1 x\left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_0^{1-x}\,\mathrm{d}x \\ &= \int_0^1 x\left(\frac{x(1-x)^2}{2}+\frac{(1-x)^3}{3}\right)\,\mathrm{d}x \\ &= \frac{1}{30}.\end{aligned}$$

3.5.29 For $t\in[0;1]$, first argue that $\int_0^1 f(x)\mathrm{d}x \geq (1-t)f(t) \geq f(t)-t$. Hence
$$\int_0^1\int_0^1 (f(x)\mathrm{d}x)\,\mathrm{d}y \geq \int_0^1 (f\circ g)(y)\mathrm{d}y - \int_0^1 g(y)\mathrm{d}y.$$
Since $\int_0^1\int_0^1 f(x)\mathrm{d}x\mathrm{d}y = \int_0^1 f(x)\mathrm{d}x$, the desired inequality is established.

3.5.30 Put $f(x,y)=xy+y^2$. If I, II, III, IV stand for the intersection of the region with each quadrant, then
$$\int_{II} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{II} f(-x,y)\mathrm{d}(-x)\mathrm{d}y = -\int_{I} f(x,y)\mathrm{d}x\mathrm{d}y,$$

$$\int_{IV} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{IV} f(x,-y)\mathrm{d}x\mathrm{d}(-y) = -\int_{I} f(x,y)\mathrm{d}x\mathrm{d}y,$$

and

$$\int_{III} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{III} f(-x,-y)\mathrm{d}(-x)\mathrm{d}(-y) = +\int_{I} f(x,y)\mathrm{d}x\mathrm{d}y.$$

Thus

$$\begin{aligned}\int_S (xy+y^2)\mathrm{d}x\mathrm{d}y &= \int_I f(x,y)\mathrm{d}x\mathrm{d}y + \int_{II} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{III} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{IV} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_I f(x,y)\mathrm{d}x\mathrm{d}y - \int_I f(x,y)\mathrm{d}x\mathrm{d}y + \int_I f(x,y)\mathrm{d}x\mathrm{d}y - \int_I f(x,y)\mathrm{d}x\mathrm{d}y \\ &= 0.\end{aligned}$$

3.5.31 We split the rectangle $[0;a]\times[0;b]$ into two triangles, depending on whether $bx < ay$ or $bx \geq ay$. Hence

$$\begin{aligned}\int_0^a\int_0^b e^{\max(b^2x^2,a^2y^2)}\,\mathrm{d}y\mathrm{d}x &= \int_{bx<ay} e^{\max(b^2x^2,a^2y^2)}\,\mathrm{d}y\mathrm{d}x + \int_{bx\geq ay} e^{\max(b^2x^2,a^2y^2)}\,\mathrm{d}y\mathrm{d}x \\ &= \int_{bx<ay} e^{a^2y^2}\,\mathrm{d}y\mathrm{d}x + \int_{bx\geq ay} e^{b^2x^2}\,\mathrm{d}y\mathrm{d}x \\ &= \int_0^b\int_0^{ay/b} e^{a^2y^2}\mathrm{d}x\mathrm{d}y + \int_0^a\int_0^{bx/a} e^{a^2y^2}\mathrm{d}y\mathrm{d}x \\ &= \int_0^b \frac{aye^{a^2y^2}}{b}\mathrm{d}y + \int_0^a \frac{bxe^{a^2y^2}}{a}\mathrm{d}x \\ &= \frac{e^{a^2b^2}-1}{2ab} + \frac{e^{a^2b^2}-1}{2ab} \\ &= \frac{e^{a^2b^2}-1}{ab}.\end{aligned}$$

3.5.32 Observe that $x \geq \frac{1}{2}(x+y)^2 \geq 0$. Hence we may take the positive square root giving $y \leq \sqrt{2x} - x$. Since $y \geq 0$, we must have $\sqrt{2x} - x \geq 0$ which means that $x \leq 2$. The integral equals

$$\begin{aligned}\int_0^2\left(\int_0^{\sqrt{2x}-x}\sqrt{xy}\mathrm{d}y\right)\mathrm{d}x &= \frac{2}{3}\int_0^2 \sqrt{x}(\sqrt{2x}-x)^{3/2}\mathrm{d}x \\ &= \frac{4}{3}\int_0^{\sqrt{2}} u^2(u\sqrt{2}-u^2)^{3/2}\mathrm{d}u \\ &= \frac{1}{6}\int_{-1}^1 (1-v^2)^{3/2}(1+v)^2\mathrm{d}v \\ &= \frac{1}{6}\int_{-\pi/2}^{\pi/2}\cos^4\theta(1+\sin^2\theta)\mathrm{d}\theta \\ &= \frac{7\pi}{96}.\end{aligned}$$

3.5.33 Observe that

$$\int_0^a \sin 2\pi x\,\mathrm{d}x = \begin{cases} 0 & \text{if } a \text{ is an integer} \\ \dfrac{1}{2\pi}(1-\cos 2\pi a) & \text{if } a \text{ is not an integer}\end{cases}$$

Thus

$$\int_0^a \sin 2\pi x\,\mathrm{d}x = 0 \iff a \text{ is an integer.}$$

Now

$$\sum_{k=1}^N \int_{R_k} \sin 2\pi x \sin 2\pi y\,\mathrm{d}x\mathrm{d}y = 0$$

since at least one of the sides of each R_k is an integer. Since

$$\int_R \sin 2\pi x \sin 2\pi y\,\mathrm{d}x\mathrm{d}y = \sum_{k=1}^N \int_{R_k} \sin 2\pi x \sin 2\pi y\,\mathrm{d}x\mathrm{d}y,$$

we deduce that at least one of the sides of R is an integer, finishing the proof.

3.5.34 We have

$$\int_0^1\int_0^1\cdots\int_0^1(x_1x_2\cdots x_n)\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n=\prod_{k=1}^n\left(\int_0^1 x_k\mathrm{d}x_k\right)=\prod_{k=1}^n\frac{1}{2}=\frac{1}{2^n}.$$

3.5.35 This is

$$\begin{aligned}\int_0^1\int_0^1\cdots\int_0^1\left(\sum_{k=1}^n x_k\right)\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n &= \sum_{k=1}^n\int_0^1\int_0^1\cdots\int_0^1 x_k\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\\ &= \sum_{k=1}^n\frac{1}{2}\\ &= \frac{n}{2}.\end{aligned}$$

3.5.41 The integral equals

$$\begin{aligned}\int_D\frac{1}{(x+y)^4}\mathrm{d}x\mathrm{d}y &= \int_1^3\left(\int_1^{4-x}\frac{\mathrm{d}y}{(x+y)^4}\,\mathrm{d}y\right)\,\mathrm{d}x\\ &= \int_1^3\left[-\frac{1}{3}(x+y)^{-3}\right]_1^{4-x}\,\mathrm{d}x\\ &= \frac{1}{3}\int_1^3\left(\frac{1}{(1+x)^3}-\frac{1}{64}\right)\,\mathrm{d}x\\ &= \frac{1}{48}.\end{aligned}$$

3.5.45 The integral equals

$$\begin{aligned}\int_D x\mathrm{d}x\mathrm{d}y &= \int_{-1}^{2/3}\left(\int_0^{x+1}\,\mathrm{d}y\right)x\,\mathrm{d}x+\int_{2/3}^4\left(\int_0^{2-\frac{x}{2}}\,\mathrm{d}y\right)x\,\mathrm{d}x\\ &= \int_{-1}^{2/3}x(x+1)\,\mathrm{d}x+\int_{2/3}^4 x\left(2-\frac{x}{2}\right)\,\mathrm{d}x\\ &= \frac{275}{54}.\end{aligned}$$

3.5.46 Make the change of variables $x_k=1-y_k$. Then

$$I=\int_0^1\int_0^1\cdots\int_0^1\cos^2\left(\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right)\mathrm{d}x_1\,\mathrm{d}x_2\ldots\,\mathrm{d}x_n$$

equals

$$\int_0^1\int_0^1\cdots\int_0^1\sin^2\left(\frac{\pi}{2n}(y_1+y_2+\cdots+y_n)\right)\mathrm{d}y_1\,\mathrm{d}y_2\ldots\,\mathrm{d}y_n.$$

Since $\sin^2 t+\cos^2 t=1$, we have $2I=1$, and so $I=\dfrac{1}{2}$.

3.6.1 ❶ Put $x=\dfrac{u+v}{2}$ and $y=\dfrac{u-v}{2}$. Then $x+y=u$ and $x-y$. Observe that D' is the triangle in the uv plane bounded by the lines $u=0,u=1,v=u,v=-u$. Its image under Φ is the triangle bounded by the equations $x=0,y=0,x+y=1$. Clearly also

$$\mathrm{d}x\wedge\mathrm{d}y=\frac{1}{2}\mathrm{d}u\wedge\mathrm{d}v.$$

❷ From the above

$$\begin{aligned}\int_D(x+y)^2e^{x^2-y^2}\mathrm{d}A &= \frac{1}{2}\int_{D'}u^2e^{uv}\mathrm{d}u\mathrm{d}v\\ &= \frac{1}{2}\int_0^1\int_{-u}^u u^2e^{uv}\mathrm{d}u\mathrm{d}v\\ &= \frac{1}{2}\int_0^1 u(e^{u^2}-e^{-u^2})\mathrm{d}u\\ &= \frac{1}{4}(e+e^{-1}-2).\end{aligned}$$

3.6.4 Here we argue that

$$\mathrm{d}u = y\mathrm{d}x + x\mathrm{d}y,$$

$$\mathrm{d}v = -2x\mathrm{d}x + 2y\mathrm{d}y.$$

Taking the wedge product of differential forms,

$$\mathrm{d}u \wedge \mathrm{d}v = 2(y^2 + x^2)\mathrm{d}x \wedge \mathrm{d}y.$$

Hence

$$\begin{aligned} f(x,y)\mathrm{d}x \wedge \mathrm{d}y &= (y^4 - x^4)\frac{1}{2(y^2+x^2)}\mathrm{d}u \wedge \mathrm{d}v \\ &= \frac{1}{2}(y^2 - x^2)\mathrm{d}u \wedge \mathrm{d}v \\ &= \frac{v}{2}\mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

The region transforms into

$$\Delta = [a; b] \times [0; 1].$$

The integral becomes

$$\begin{aligned} \int_D f(x,y)\mathrm{d}x \wedge \mathrm{d}y &= \int_\Delta v\ \mathrm{d}u \wedge \mathrm{d}v \\ &= \frac{1}{2}\left(\int_a^b \mathrm{d}u\right)\left(\int_0^1 v\ \mathrm{d}v\right) \\ &= \frac{b-a}{4}. \end{aligned}$$

3.6.5 ❶ Formally,

$$\begin{aligned} \int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} &= \int_0^1 \int_0^1 (1 + xy + x^2y^2 + x^3y^3 + \cdots)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \left(y + \frac{xy^2}{2} + \frac{x^2y^3}{3} + \frac{x^3y^4}{4} + \cdots\right)_0^1 \mathrm{d}x \\ &= \int_0^1 (1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots)\mathrm{d}x \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \end{aligned}$$

❷ This change of variables transforms the square $[0;1] \times [0;1]$ in the xy plane into the square with vertices at $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$ in the uv plane. We will split this region of integration into two disjoint triangles: T_1 with vertices at $(0,0)$, $(1,1)$, $(1,-1)$, and T_2 with vertices at $(1,-1)$, $(1,1)$, $(2,0)$. Observe that

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2}\mathrm{d}u \wedge \mathrm{d}v,$$

and that $u + v = 2x, u - v = 2y$ and so $4xy = u^2 - v^2$. The integral becomes

$$\begin{aligned} \int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} &= \frac{1}{2}\int_{T_1 \cup T_2} \frac{\mathrm{d}u \wedge \mathrm{d}v}{1 - \frac{u^2-v^2}{4}} \\ &= 2\int_0^1 \left(\int_{-u}^u \frac{\mathrm{d}v}{4 - u^2 + v^2}\right)\mathrm{d}u + 2\int_1^2 \left(\int_{u-2}^{2-u} \frac{\mathrm{d}v}{4 - u^2 + v^2}\right)\mathrm{d}u, \end{aligned}$$

as desired.

❸ This follows by using the identity

$$\int_0^t \frac{\mathrm{d}\omega}{1 + \Omega^2} = \arctan t.$$

❹ This is straightforward but tedious!

3.7.1 The integral in Cartesian coordinates is

$$\begin{aligned} \int_1^{\sqrt{15}} \int_1^{\sqrt{16-y^2}} xy\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int \int_1^{\sqrt{15}} 15y - y^3\mathrm{d}y \\ &= \frac{49}{2}. \end{aligned}$$

The integral in polar coordinates is

$$\begin{aligned}\int_{\arcsin\frac{1}{4}}^{\frac{\pi}{4}}\int_{1/\sin\theta}^{4} r^3\sin\theta\cos\theta \mathrm{d}r\mathrm{d}\theta+\int_{\frac{\pi}{4}}^{\arccos\frac{1}{4}}\int_{1/\cos\theta}^{4} r^3\sin\theta\cos\theta \mathrm{d}r\mathrm{d}\theta &= \frac{1}{4}\int_{\arcsin\frac{1}{4}}^{\frac{\pi}{4}}\left(4^4-\frac{1}{\sin^4\theta}\right)\sin\theta\cos\theta\mathrm{d}\theta \\ &\quad +\frac{1}{4}\int_{\frac{\pi}{4}}^{\arccos\frac{1}{4}}\left(4^4-\frac{1}{\cos^4\theta}\right)\sin\theta\cos\theta\mathrm{d}\theta \\ &= \frac{4^4}{4}\int_{\arcsin\frac{\pi}{4}}^{\arccos\frac{1}{4}}\sin\theta\cos\theta\mathrm{d}\theta \\ &\quad -\frac{1}{4}\int_{\arcsin\frac{1}{4}}^{\frac{\pi}{4}}(\cot\theta)(\csc^2\theta)\mathrm{d}\theta \\ &\quad -\frac{1}{4}\int_{\frac{\pi}{4}}^{\arccos\frac{1}{4}}(\tan\theta)(\sec^2\theta)\mathrm{d}\theta \\ &= 28-\frac{7}{4}-\frac{7}{4} \\ &= \frac{49}{2}\end{aligned}$$

3.7.2 Using polar coordinates,

$$\begin{aligned}\int_D x^2-y^2\mathrm{d}x\mathrm{d}y &= \int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos\theta}\rho^3\ \mathrm{d}\rho\right)(\cos^2\theta-\sin^2\theta)\mathrm{d}\theta \\ &= 8\int_0^{\pi/2}\cos^4\theta(\cos^2\theta-\sin^2\theta)\ \mathrm{d}\theta \\ &= \pi.\end{aligned}$$

3.7.3 Using polar coordinates,

$$\begin{aligned}\int_D \sqrt{xy}\mathrm{d}x\mathrm{d}y &= 4\int_0^{\pi/4}\left(\int_0^{\sqrt{\sin 2\theta}}\rho\sqrt{\rho^2\cos\theta\sin\theta}\ \mathrm{d}\rho\right)\mathrm{d}\theta \\ &= \frac{4}{3}\int_0^{\pi/4}(\sqrt{\sin 2\theta})^3\sqrt{\cos\theta\sin\theta}\ \mathrm{d}\theta \\ &= \frac{4}{3\sqrt{2}}\int_0^{\pi/4}\sin^2 2\theta\ \mathrm{d}\theta \\ &= \frac{\pi\sqrt{2}}{12}.\end{aligned}$$

3.7.4 Using $x=a\rho\cos\theta$, $y=b\rho\sin\theta$, the integral becomes

$$(ab)\left(\int_0^{2\pi}a^3\cos^3\theta+b^3\sin^3\theta\mathrm{d}\theta\right)\left(\int_0^1\rho^4\mathrm{d}\rho\right)=\frac{2}{15}(ab)(a^3+b^3).$$

3.7.8 Using polar coordinates,

$$\begin{aligned}\int_D f(x,y)\mathrm{d}A &= \int_0^{\pi/6}\left(\int_{2\sin\theta}^1\rho^2\ \mathrm{d}\rho\right)\mathrm{d}\theta \\ &= \frac{1}{3}\int_0^{\pi/6}(1-8\sin^3\theta)\ \mathrm{d}\theta \\ &= \frac{\pi}{18}-\frac{16}{9}+\sqrt{3}.\end{aligned}$$

3.7.9 Using polar coordinates the integral becomes

$$\int_0^{\pi/2}\left(\int_0^{2\cos\theta}\rho^4\mathrm{d}\rho\right)\cos^2\theta\sin\theta\mathrm{d}\theta=\frac{4}{5}.$$

3.7.11 Using polar coordinates the integral becomes

$$\int_{-\pi/4}^{\pi/4}\left(\int_{1/\cos\theta}^{2\cos\theta}\frac{1}{\rho^3}\mathrm{d}\rho\right)\mathrm{d}\theta=\int_0^{\pi/4}\left(\cos^2\theta-\frac{\sec^2\theta}{4}\right)\mathrm{d}\theta=\frac{\pi}{8}.$$

3.7.12 Put
$$D' = \{(x,y) \in \mathbb{R}^2 : y \geq x, x^2+y^2-y \leq 0, x^2+y^2-x \leq 0\}.$$
Then the integral equals
$$2\int_{D'} (x+y)^2 \mathrm{d}x\mathrm{d}y.$$
Using polar coordinates the integral equals
$$\begin{aligned} 2\int_{\pi/4}^{\pi/2} (\cos\theta+\sin\theta)^2 \left(\int_0^{\cos\theta} \rho^3 \mathrm{d}\rho\right)\mathrm{d}\theta &= \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^4\theta(1+2\sin\theta\cos\theta)\mathrm{d}\theta \\ &= \frac{3\pi}{64} - \frac{5}{48}. \end{aligned}$$

3.7.13 Observe that $D = D_2 \setminus D_1$ where D_2 is the disk limited by the equation $x^2+y^2 = 1$ and D_1 is the disk limited by the equation $x^2+y^2 = y$. Hence
$$\int_D \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2} = \int_{D_2} \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2} - \int_{D_1} \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2}.$$
Using polar coordinates we have
$$\int_{D_2} \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2} = \int_0^{2\pi}\int_0^1 \frac{\rho}{(1+\rho^2)^2}\mathrm{d}\rho\mathrm{d}\theta = \frac{\pi}{2}$$
and
$$\begin{aligned} \int_{D_1} \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2} &= 2\int_0^{\pi/2}\int_0^{\sin\theta} \frac{\rho}{(1+\rho^2)^2}\mathrm{d}\rho\mathrm{d}\theta = \int_0^{\pi/2} \frac{\sin^2\theta\mathrm{d}\theta}{1+\sin^2\theta} \\ &= \int_0^{+\infty} \frac{\mathrm{d}t}{t^2+1} - \frac{\mathrm{d}t}{2t^2+1} = \frac{\pi}{2} - \frac{\pi\sqrt{2}}{4}. \end{aligned}$$
(We evaluated this last integral using $t = \tan\theta$) Finally, the integral equals
$$\frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{\pi\sqrt{2}}{4}\right) = \frac{\pi\sqrt{2}}{4}.$$

3.7.14 We have
$$2x\mathrm{d}x = \cos\theta\mathrm{d}\rho - \rho\sin\theta\mathrm{d}\theta, \quad 2y\mathrm{d}y = \sin\theta\mathrm{d}\rho + \rho\cos\theta\mathrm{d}\theta,$$
whence
$$4xy\mathrm{d}x \wedge \mathrm{d}y = \rho\mathrm{d}\rho \wedge \mathrm{d}\theta.$$
It follows that
$$\begin{aligned} x^3y^3\sqrt{1-x^4-y^4}\, \mathrm{d}x \wedge \mathrm{d}y &= \frac{1}{4}(x^2y^2)(\sqrt{1-x^4-y^4})(4xy\, \mathrm{d}x \wedge \mathrm{d}y) \\ &= \frac{1}{4}(\rho^3\cos\theta\sin\theta\sqrt{1-\rho^2})\mathrm{d}\rho \wedge \mathrm{d}\theta \end{aligned}$$
Observe that
$$x^4+y^4 \leq 1 \implies \rho^2\cos^2\theta + \rho^2\sin^2\theta \leq 1 \implies \rho \leq 1.$$
Since the integration takes place on the first quadrant, we have $0 \leq \theta \leq \pi/2$. Hence the integral becomes
$$\begin{aligned} \int_0^{\pi/2}\int_0^1 \frac{1}{4}(\rho^3\cos\theta\sin\theta\sqrt{1-\rho^2})\mathrm{d}\rho\mathrm{d}\theta &= \frac{1}{4}\left(\int_0^{\pi/2}\cos\theta\sin\theta\mathrm{d}\theta\right)\left(\int_0^1 \rho^3\sqrt{1-\rho^2}\mathrm{d}\rho\right) \\ &= \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{2}{15} \\ &= \frac{1}{60}. \end{aligned}$$

3.7.15 ❶ Using polar coordinates
$$I_a = \int_0^{2\pi}\left(\int_0^a \rho e^{-\rho^2}\mathrm{d}\rho\right)\mathrm{d}\theta = \pi(1-e^{-a^2}).$$

❷ The domain of integration of J_a is a square of side $2a$ centred at the origin. The respective domains of integration of I_a and $I_{a\sqrt{2}}$ are the inscribed and the exscribed circles to the square.

❸ First observe that
$$J_a = \left(\int_{-a}^a e^{-x^2}\mathrm{d}x\right)^2.$$
Since both I_a and $I_{a\sqrt{2}}$ tend to π as $a \to +\infty$, we deduce that $J_a \to \pi$. This gives the result.

3.7.16

$$\begin{aligned}
\int_{4\leq x^2+y^2\leq 16} \frac{1}{x^2+xy+y^2}\mathrm{d}A &= \int_0^{2\pi}\int_2^4 \frac{r}{r^2+r^2\sin\theta\cos\theta}\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_2^4 \frac{1}{r(1+\sin\theta\cos\theta)}\mathrm{d}r\mathrm{d}\theta \\
&= \left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{1+\sin\theta\cos\theta}\right)\left(\int_2^4\frac{\mathrm{d}r}{r}\right) \\
&= \left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{1+\sin\theta\cos\theta}\right)\log 2 \\
&= 2\left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta}\right)\log 2 \\
&= 4\left(\int_0^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta}\right)\log 2 \\
&= 4I\log 2,
\end{aligned}$$

so the problem reduces to evaluate $I = \displaystyle\int_0^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta}$. To find this integral, we now use what has been dubbed as "the world's sneakiest substitution"[1]: we put $\tan\theta = t$. In so doing we have to pay attention to the fact that $\theta\mapsto\tan\theta$ is not continuous on $[0;\pi]$, so we split the interval of integration into two pieces, $[0;\pi] = [0;\frac{\pi}{2}]\cup]\frac{\pi}{2};\pi]$. Then $\sin 2\theta = \dfrac{2t}{1+t^2}$, $\cos 2\theta = \dfrac{1-t^2}{1+t^2}$, $\mathrm{d}\theta = \dfrac{\mathrm{d}t}{1+t^2}$. Hence

$$\begin{aligned}
\int_0^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta} &= \int_0^{\pi/2}\frac{\mathrm{d}\theta}{2+\sin 2\theta} + \int_{\pi/2}^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta} \\
&= \int_0^{+\infty}\frac{\frac{\mathrm{d}t}{1+t^2}}{2+\frac{2t}{1+t^2}} + \int_{-\infty}^{0}\frac{\frac{\mathrm{d}t}{1+t^2}}{2+\frac{2t}{1+t^2}} \\
&= \int_0^{+\infty}\frac{\mathrm{d}t}{2(t^2+t+1)} + \int_{-\infty}^{0}\frac{\mathrm{d}t}{2(t^2+t+1)} \\
&= \frac{2}{3}\int_0^{+\infty}\frac{\mathrm{d}t}{(\frac{2t}{\sqrt{3}}+\frac{1}{\sqrt{3}})^2+1} + \frac{2}{3}\int_{-\infty}^{0}\frac{\mathrm{d}t}{(\frac{2t}{\sqrt{3}}+\frac{1}{\sqrt{3}})^2+1} \\
&= \frac{\sqrt{3}}{3}\Big|_0^{+\infty}\arctan\left(\frac{2t\sqrt{3}}{3}+\frac{\sqrt{3}}{3}\right) + \frac{\sqrt{3}}{3}\Big|_{-\infty}^{0}\arctan\left(\frac{2t\sqrt{3}}{3}+\frac{\sqrt{3}}{3}\right) \\
&= \frac{\sqrt{3}}{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) + \frac{\sqrt{3}}{3}\left(\frac{\pi}{6}-\left(-\frac{\pi}{2}\right)\right) \\
&= \frac{\pi\sqrt{3}}{3}.
\end{aligned}$$

We conclude that

$$\int_{4\leq x^2+y^2\leq 16}\frac{1}{x^2+xy+y^2}\mathrm{d}A = \frac{4\pi\sqrt{3}\log 2}{3}.$$

3.7.17 Recall from formula 1.14 that the area enclosed by a simple closed curve Γ is given by

$$\frac{1}{2}\int_{\Gamma} x\mathrm{d}y - y\mathrm{d}x.$$

Using polar coordinates

$$\begin{aligned}
x\mathrm{d}y - y\mathrm{d}x &= (\rho\cos\theta)(\sin\theta\mathrm{d}\rho + \rho\cos\theta\mathrm{d}\theta) - (\rho\sin\theta)(\cos\theta\mathrm{d}\rho - \rho\sin\theta\mathrm{d}\theta) \\
&= \rho^2\mathrm{d}\theta.
\end{aligned}$$

Parametrise the curve enclosing the region by polar coordinates so that the region is tangent to the polar axis at the origin. Let the equation of the curve be $\rho = f(\theta)$. The area of the region is then given by

$$\frac{1}{2}\int_0^{\pi}\rho^2\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi}(f(\theta))^2\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/2}((f(\theta))^2 + (f(\theta+\pi/2))^2)\mathrm{d}\theta.$$

By the Pythagorean Theorem, the integral above is the integral of the square of the chord in question. If no two points are farther than 2 units, their squares are no farther than 4 units, and so the area

$$< \frac{1}{2}\int_0^{\pi/2} 4\mathrm{d}\theta = \pi,$$

[1] by Michael Spivak, whose *Calculus* book I recommend greatly.

a contradiction.

3.7.18 Let $I(S)$ denote the integral sought over a region S. Since $D(x,y)=0$ inside R, $I(R)=A$. Let $\mathscr{L}$ be a side of R with length l and let $S(\mathscr{L})$ be the half strip consisting of the points of the plane having a point on $\mathscr{L}$ as nearest point of R. Set up coordinates uv so that u is measured parallel to $\mathscr{L}$ and v is measured perpendicular to L. Then

$$I(S(\mathscr{L}))=\int_0^l\int_0^{+\infty}e^{-v}\,\mathrm{d}u\mathrm{d}v=l.$$

The sum of these integrals over all the sides of R is L.

If $\mathscr{V}$ is a vertex of R, the points that have $\mathscr{V}$ as nearest from R lie inside an angle $S(\mathscr{V})$ bounded by the rays from $\mathscr{V}$ perpendicular to the edges meeting at $\mathscr{V}$. If α is the measure of that angle, then using polar coordinates

$$I(S(\mathscr{V}))=\int_0^\alpha\int_0^{+\infty}\rho e^{-\rho}\,\mathrm{d}\rho\mathrm{d}\theta=\alpha.$$

The sum of these integrals over all the vertices of R is 2π. Assembling all these integrals we deduce the result.

3.8.1 We have

$$\begin{aligned}\int_E z\mathrm{d}V &= \int_0^1\int_0^{1-y}\int_0^{1-z} z\mathrm{d}x\mathrm{d}z\mathrm{d}y\\ &= \int_0^1\int_0^{1-y} z-z^2\mathrm{d}z\mathrm{d}y\\ &= \int_0^1\frac{(1-y)^2}{2}-\frac{(1-y)^3}{3}\mathrm{d}y\\ &= \frac{(1-y)^4}{12}-\frac{(1-y)^3}{6}\Big|_0^1\\ &= =\frac{1}{6}.\end{aligned}$$

3.8.3 Let $A=(1,1,1)$, $B=(1,0,0)$, $C=(0,0,1)$, and $O=(0,0,0)$. We have four planes passing through each triplet of points:

$$\begin{array}{lll} P_1: & A,B,C, & x-y+z=1\\ P_2: & A,B,O & z=y\\ P_3: & A,C,O & x=y\\ P_4: & B,C,O & y=0.\end{array}$$

Using the order of integration $\mathrm{d}z\mathrm{d}x\mathrm{d}y$, z sweeps from P_2 to P_1, so the limits are $z=y$ to $z=1-x+y$. The projection of the solid on the xy plane produces the region bounded by the lines $x=0$, $x=1$ and $x=y$ on the first quadrant of the xy-plane. Thus

$$\begin{aligned}\int_0^1\int_0^x\int_y^{1-x+y}\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_0^1\int_0^x(1-x)\mathrm{d}y\mathrm{d}x\\ &= \int_0^1\left(x-x^2\right)\mathrm{d}x\\ &= \left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1\\ &= \frac{1}{6}.\end{aligned}$$

We use the same limits of integration as in the previous integral. We have

$$\begin{aligned}\int_0^1\int_0^x\int_y^{1-x+y}x\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_0^1\int_0^x(x-x^2)\mathrm{d}y\mathrm{d}x\\ &= \int_0^1\left(x^2-x^3\right)\mathrm{d}x\\ &= \left(\frac{x^3}{3}-\frac{x^4}{4}\right)\Big|_0^1\\ &= \frac{1}{12}.\end{aligned}$$

3.8.4 We have

$$\int_E x\mathrm{d}V=\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{y/3}x\mathrm{d}z\mathrm{d}y\mathrm{d}x=\frac{27}{8}.$$

3.8.5 The desired integral is

$$\begin{aligned}
\int_0^1\int_0^1\int_0^{\infty}\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1+x^2z^2)(1+y^2z^2)} &= \int_0^1\int_0^1\int_0^{\infty}\frac{1}{x^2-y^2}\left(\frac{x^2}{1+x^2z^2}-\frac{y^2}{1+y^2z^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^1\int_0^1\frac{1}{x^2-y^2}\left(x\arctan(xz)-y\arctan(yz)\right)\Big|_{z=0}^{z=\infty}\mathrm{d}x\mathrm{d}y \\
&= \int_0^1\int_0^1\frac{\pi(x-y)}{2(x^2-y^2)}\mathrm{d}x\mathrm{d}y \\
&= \int_0^1\int_0^1\frac{\pi}{2(x+y)}\mathrm{d}x\mathrm{d}y \\
&= \frac{\pi}{2}\int_0^1\log(y+1)-\log y\mathrm{d}y \\
&= \frac{\pi}{2}\cdot\left((y+1)\log(y+1)-(y+1)-y\log y+y\right)\Big|_0^1 \\
&= \pi\log 2.
\end{aligned}$$

3.9.1 Cartesian:

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}}\mathrm{d}z\mathrm{d}x\mathrm{d}y.$$

Cylindrical:

$$\int_0^1\int_0^{2\pi}\int_{r^2}^{r}r\mathrm{d}z\mathrm{d}\theta\mathrm{d}r.$$

Spherical:

$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^{(\cos\phi)/(\sin\phi)^2}r^2\sin\phi\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

The volume is $\frac{\pi}{3}$.

3.9.2

1. Since $x^2+y^2\leq z\leq\sqrt{4-x^2-y^2}$, we start our integration with the z-variable. Observe that if (x,y,z) is on the intersection of the surfaces then

$$z^2+z=4 \implies z=\frac{-1\pm\sqrt{17}}{2}.$$

Since $x^2+y^2+z^2=4 \implies -2\leq z\leq 2$, we must have $z=\frac{\sqrt{17}-1}{2}$ only. The projection of the circle of intersection of the paraboloid and the sphere onto the xy-plane satisfies the equation

$$z^2+z=4 \implies x^2+y^2+(x^2+y^2)^2=4 \implies x^2+y^2=\frac{\sqrt{17}-1}{2},$$

a circle of radius $\sqrt{\frac{\sqrt{17}-1}{2}}$. The desired integral is thus

$$\int_{-\sqrt{\frac{\sqrt{17}-1}{2}}}^{\sqrt{\frac{\sqrt{17}-1}{2}}}\int_{-\sqrt{\frac{\sqrt{17}-1}{2}-x^2}}^{\sqrt{\frac{\sqrt{17}-1}{2}-x^2}}\int_{x^2+y^2}^{\sqrt{4-x^2-y^2}}x\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

2. The z-limits remain the same as in the Cartesian coordinates, but translated into cylindrical coordinates, and so $r^2\leq z\leq\sqrt{4-r^2}$. The projection of the intersection circle onto the xy-plane is again a circle with centre at the origin and radius $\sqrt{\frac{\sqrt{17}-1}{2}}$. The desired integral

$$\int_0^{2\pi}\int_0^{\sqrt{\frac{\sqrt{17}-1}{2}}}\int_{r^2}^{\sqrt{4-r^2}}r^2\cos\theta\mathrm{d}z\mathrm{d}r\mathrm{d}\theta.$$

3. Observe that

$$z = x^2 + y^2 \implies r\cos\phi = r^2(\cos\theta)^2(\sin\phi)^2 + r^2(\sin\theta)^2(\sin\phi)^2 \implies r \in \{0, (\csc\phi)(\cot\phi)\}.$$

It is clear that the limits of the angle θ are from $\theta = 0$ to $\theta = 2\pi$. The angle ϕ starts at $\phi = 0$. Now,

$$z = r\cos\phi \implies \cos\phi = \frac{\frac{\sqrt{17}-1}{2}}{2} \implies \phi = \arccos\left(\frac{\sqrt{17}-1}{4}\right)$$

The desired integral

$$\int_0^{2\pi}\int_0^{\arccos\left(\frac{\sqrt{17}-1}{4}\right)}\int_{(\csc\phi)(\cot\phi)}^{2} r^3\cos\theta\sin^2\phi \mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta.$$

Perhaps it is easiest to evaluate the integral using cylindrical coordinates. We obtain

$$\int_0^{2\pi}\int_0^{\sqrt{\frac{\sqrt{17}-1}{2}}}\int_{r^2}^{\sqrt{4-r^2}} r^2\cos\theta \mathrm{d}z\mathrm{d}r\mathrm{d}\theta = 0,$$

a conclusion that is easily reached, since the integrand is an odd function of x and the domain of integration is symmetric about the origin in x.

3.9.3 Cartesian:

$$\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_1^{\sqrt{4-x^2-y^2}} \mathrm{d}z\mathrm{d}x\mathrm{d}y.$$

Cylindrical:

$$\int_0^{\sqrt{3}}\int_0^{2\pi}\int_1^{\sqrt{4-r^2}} r\mathrm{d}z\mathrm{d}\theta\mathrm{d}r.$$

Spherical:

$$\int_0^{\pi/3}\int_0^{2\pi}\int_{1/\cos\phi}^{2} r^2\sin\phi \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

The volume is $\frac{5\pi}{3}$.

3.9.5 We have

$$\int_E y\mathrm{d}V = \int_0^{2\pi}\int_1^2\int_0^{2+r\cos\theta} r^2\sin\theta \mathrm{d}z\mathrm{d}r\mathrm{d}\theta = 0.$$

3.9.7 $\frac{\pi}{96}$

3.9.8 $\frac{\pi}{14}$

3.9.9 We put

$$x = \rho\cos\theta\sin\phi\sin t; y = \rho\sin\theta\sin\phi\sin t; u = \rho\cos\phi\sin t; v = \rho\cos t.$$

Upon using $\sin^2 a + \cos^2 a = 1$ three times,

$$\begin{aligned} x^2+y^2+u^2+v^2 &= r^2\cos^2\theta\sin^2\phi\sin^2 t + r^2\sin^2\theta\sin^2\phi\sin^2 t + r^2\cos^2\phi\sin^2 t + r^2\cos^2 t \\ &= r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\phi \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2. \end{aligned}$$

Now,

$$\begin{aligned} \mathrm{d}x &= \cos\theta\sin\phi\sin t\mathrm{d}r - \rho\sin\theta\sin\phi\sin t\mathrm{d}\theta + \rho\cos\theta\cos\phi\sin t\mathrm{d}\phi + \rho\cos\theta\sin\phi\cos t\mathrm{d}t \\ \mathrm{d}y &= \sin\theta\sin\phi\sin t\mathrm{d}r + \rho\cos\theta\sin\phi\sin t\mathrm{d}\theta + \rho\sin\theta\cos\phi\sin t\mathrm{d}\phi + \rho\sin\theta\sin\phi\cos t\mathrm{d}t \\ \mathrm{d}u &= \cos\phi\sin t\mathrm{d}r - \rho\sin\phi\sin t\mathrm{d}\phi + \rho\cos\phi\cos t\mathrm{d}t \\ \mathrm{d}v &= \cos t\mathrm{d}r - \rho\sin t\mathrm{d}t \end{aligned}.$$

After some calculation,

$$\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}u \wedge \mathbf{d}v = r^3 \sin\phi \sin^2 t \mathbf{d}r \wedge \mathbf{d}\phi \wedge \mathbf{d}\theta \wedge \mathbf{d}t.$$

Therefore

$$\begin{aligned}\iiiint\limits_{x^2+y^2+u^2+v^2\leq 1} e^{x^2+y^2+u^2+v^2}\mathbf{d}x\mathbf{d}y\mathbf{d}u\mathbf{d}v &= \int_0^{\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1 r^3 e^{r^2}\sin\phi\sin^2 t\mathbf{d}r\mathbf{d}\phi\mathbf{d}\theta\mathbf{d}t \\ &= \left(\int_0^1 r^3 e^{r^2}\mathbf{d}r\right)\left(\int_0^{2\pi}\mathbf{d}\theta\right)\left(\int_0^{\pi}\sin\phi\mathbf{d}\phi\right)\left(\int_0^{\pi}\sin^2 t\mathbf{d}t\right) \\ &= \left(\frac{1}{2}\right)(2\pi)(2)\left(\frac{\pi}{2}\right) \\ &= \pi^2.\end{aligned}$$

3.9.10 We make the change of variables

$$u = x+y+z \implies \mathbf{d}u = \mathbf{d}x+\mathbf{d}y+\mathbf{d}z,$$

$$uv = y+z \implies u\mathbf{d}v + v\mathbf{d}u = \mathbf{d}y+\mathbf{d}z,$$

$$uvw = z \implies uv\mathbf{d}w + uw\mathbf{d}v + vw\mathbf{d}u = \mathbf{d}z.$$

This gives

$$\begin{aligned}x &= u(1-v),\\ y &= uv(1-w),\\ z &= uvw,\\ u^2v\ \mathbf{d}u\wedge\mathbf{d}v\wedge\mathbf{d}w &= \mathbf{d}x\wedge\mathbf{d}y\wedge\mathbf{d}z.\end{aligned}$$

To find the limits of integration we observe that the limits of integration using $\mathbf{d}x\wedge\mathbf{d}y\wedge\mathbf{d}z$ are

$$0\leq z\leq 1,\ 0\leq y\leq 1-z,\ 0\leq x\leq 1-y-z.$$

This translates into

$$0\leq uvw\leq 1,\ 0\leq uv-uvw\leq 1-uvw,\ 0\leq u-uv\leq 1-uv+uvw-uvw.$$

Thus

$$0\leq uvw\leq 1,\ 0\leq uv\leq 1,\ 0\leq u\leq 1,$$

which finally give

$$0\leq u\leq 1,\ 0\leq v\leq 1,\ 0\leq w\leq 1.$$

The integral sought is then, using the fact that for positive integers m, n one has

$$\int_0^1 x^m(1-x)^n\ \mathbf{d}x = \frac{m!n!}{(m+n+1)!},$$

we deduce,

$$\int_0^1\int_0^1\int_0^1 u^{20}v^{18}w^8(1-u)^4(1-v)(1-w)^9\ \mathbf{d}u\mathbf{d}v\mathbf{d}w,$$

which in turn is

$$\left(\int_0^1 u^{20}(1-u)^4\mathbf{d}u\right)\left(\int_0^1 v^{18}(1-v)\mathbf{d}v\right)\left(\int_0^1 w^8(1-w)^9\mathbf{d}w\right) = \frac{1}{265650}\cdot\frac{1}{380}\cdot\frac{1}{437580}$$

which is

$$= \frac{1}{44172388260000}.$$

3.10.1 We parametrise the surface by letting $x = u, y = v, z = u + v^2$. Observe that the domain D of Σ is the square $[0;1]\times[0;2]$. Observe that

$$\mathbf{d}x\wedge\mathbf{d}y = \mathbf{d}u\wedge\mathbf{d}v,$$

$$\mathbf{d}y\wedge\mathbf{d}z = -\mathbf{d}u\wedge\mathbf{d}v,$$

$$\mathbf{d}z\wedge\mathbf{d}x = -2v\mathbf{d}u\wedge\mathbf{d}v,$$

and so

$$\left|\left|\mathbf{d}^2\mathbf{x}\right|\right| = \sqrt{2+4v^2}\mathbf{d}u\wedge\mathbf{d}v.$$

The integral becomes

$$\begin{aligned}\int_{\Sigma} y\left|\left|\mathrm{d}^2\mathrm{x}\right|\right| &= \int_0^2\int_0^1 v\sqrt{2+4v^2}\mathrm{d}u\mathrm{d}v \\ &= \left(\int_0^1 \mathrm{d}u\right)\left(\int_0^2 y\sqrt{2+4v^2}\mathrm{d}v\right) \\ &= \frac{13\sqrt{2}}{3}.\end{aligned}$$

3.10.2 Using $x = r\cos\theta$, $y = r\sin\theta$, $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, the surface area is

$$\sqrt{2}\int_0^{2\pi}\int_1^2 r\mathrm{d}r\mathrm{d}\theta = 3\pi\sqrt{2}.$$

3.10.3 We use spherical coordinates, $(x, y, z) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Here $\theta \in [0; 2\pi]$ is the latitude and $\phi \in [0; \pi]$ is the longitude. Observe that

$$\mathrm{d}x \wedge \mathrm{d}y = \sin\phi\cos\phi\mathrm{d}\phi \wedge \mathrm{d}\theta,$$

$$\mathrm{d}y \wedge \mathrm{d}z = \cos\theta\sin^2\phi\mathrm{d}\phi \wedge \mathrm{d}\theta,$$

$$\mathrm{d}z \wedge \mathrm{d}x = -\sin\theta\sin^2\phi\mathrm{d}\phi \wedge \mathrm{d}\theta,$$

and so

$$\left|\left|\mathrm{d}^2\mathrm{x}\right|\right| = \sin\phi\mathrm{d}\phi \wedge \mathrm{d}\theta.$$

The integral becomes

$$\begin{aligned}\int_{\Sigma} x^2\left|\left|\mathrm{d}^2\mathrm{x}\right|\right| &= \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\phi\mathrm{d}\phi\mathrm{d}\theta \\ &= \frac{4\pi}{3}.\end{aligned}$$

3.10.4 Put $x = u, y = v, z^2 = u^2 + v^2$. Then

$$\mathrm{d}x = \mathrm{d}u,\ \mathrm{d}y = \mathrm{d}v,\ z\mathrm{d}z = u\mathrm{d}u + v\mathrm{d}v,$$

whence

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}u \wedge \mathrm{d}v, \mathrm{d}y \wedge \mathrm{d}z = -\frac{u}{z}\mathrm{d}u \wedge \mathrm{d}v, \mathrm{d}z \wedge \mathrm{d}x = -\frac{v}{z}\mathrm{d}u \wedge \mathrm{d}v,$$

and so

$$\begin{aligned}\left|\left|\mathrm{d}^2\mathrm{x}\right|\right| &= \sqrt{(\mathrm{d}x \wedge \mathrm{d}y)^2 + (\mathrm{d}z \wedge \mathrm{d}x)^2 + (\mathrm{d}y \wedge \mathrm{d}z)^2} \\ &= \sqrt{1 + \frac{u^2+v^2}{z^2}}\ \mathrm{d}u \wedge \mathrm{d}v \\ &= \sqrt{2}\ \mathrm{d}u \wedge \mathrm{d}v.\end{aligned}$$

Hence

$$\int_{\Sigma} z\left|\left|\mathrm{d}^2\mathrm{x}\right|\right| = \int_{u^2+v^2\leq 1}\sqrt{u^2+v^2}\ \sqrt{2}\ \mathrm{d}u\mathrm{d}v = \sqrt{2}\int_0^{2\pi}\int_0^1 \rho^2\ \mathrm{d}\rho\mathrm{d}\theta = \frac{2\pi\sqrt{2}}{3}.$$

3.10.5 If the egg has radius R, each slice will have height $2R/n$. A slice can be parametrised by $0 \leq \theta \leq 2\pi$, $\phi_1 \leq \phi \leq \phi_2$, with

$$R\cos\phi_1 - R\cos\phi_2 = 2R/n.$$

The area of the part of the surface of the sphere in slice is

$$\int_0^{2\pi}\int_{\phi_1}^{\phi_2} R^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta = 2\pi R^2(\cos\phi_1 - \cos\phi_2) = 4\pi R^2/n.$$

This means that each of the n slices has identical area $4\pi R^2/n$.

3.10.6 We project this plane onto the coordinate axes obtaining

$$\int_{\Sigma} xy\mathrm{d}y\mathrm{d}z = \int_0^6\int_0^{3-z/2}(3-y-z/2)y\mathrm{d}y\mathrm{d}z = \frac{27}{4},$$

$$-\int_{\Sigma} x^2 \mathrm{d}z\mathrm{d}x = -\int_0^3 \int_0^{6-2x} x^2 \mathrm{d}z\mathrm{d}x = -\frac{27}{2},$$

$$\int_{\Sigma} (x+z)\mathrm{d}x\mathrm{d}y = \int_0^3 \int_0^{3-y} (6-x-2y)\mathrm{d}x\mathrm{d}y = \frac{27}{2},$$

and hence

$$\int_{\Sigma} xy\mathrm{d}y\mathrm{d}z - x^2\mathrm{d}z\mathrm{d}x + (x+z)\mathrm{d}x\mathrm{d}y = \frac{27}{4}.$$

3.11.1 Evaluating this directly would result in evaluating four path integrals, one for each side of the square. We will use Green's Theorem. We have

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}(x^3y) \wedge \mathrm{d}x + \mathrm{d}(xy) \wedge \mathrm{d}y \\ &= (3x^2y\mathrm{d}x + x^3\mathrm{d}y) \wedge \mathrm{d}x + (y\mathrm{d}x + x\mathrm{d}y) \wedge \mathrm{d}y \\ &= (y - x^3)\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

The region M is the area enclosed by the square. The integral equals

$$\begin{aligned} \oint_C x^3y\mathrm{d}x + xy\mathrm{d}y &= \int_0^2 \int_0^2 (y - x^3)\mathrm{d}x\mathrm{d}y \\ &= -4. \end{aligned}$$

3.11.2 We have

❶ L_{AB} is $y = x$; L_{AC} is $y = -x$, and L_{BC} is clearly $y = -\frac{1}{3}x + \frac{4}{3}$.

❷ We have

$$\begin{aligned} \int_{AB} y^2\mathrm{d}x + x\mathrm{d}y &= \int_0^1 (x^2 + x)\mathrm{d}x &&= \frac{5}{6} \\ \int_{BC} y^2\mathrm{d}x + x\mathrm{d}y &= \int_1^{-2} \left(\left(-\frac{1}{3}x + \frac{4}{3}\right)^2 - \frac{1}{3}x\right)\mathrm{d}x &&= -\frac{15}{2} \\ \int_{CA} y^2\mathrm{d}x + x\mathrm{d}y &= \int_{-2}^0 (x^2 - x)\mathrm{d}x &&= \frac{14}{3} \end{aligned}$$

Adding these integrals we find

$$\oint_{\triangle} y^2\mathrm{d}x + x\mathrm{d}y = -2.$$

❸ We have

$$\begin{aligned} \int_{\mathscr{D}} (1-2y)\mathrm{d}x \wedge \mathrm{d}y &= \int_{-2}^0 \left(\int_{-x}^{-x/3+4/3} (1-2y)\mathrm{d}y\right)\mathrm{d}x \\ &\quad + \int_0^1 \left(\int_x^{-x/3+4/3} (1-2y)\mathrm{d}y\right)\mathrm{d}x \\ &= -\frac{44}{27} - \frac{10}{27} \\ &= -2. \end{aligned}$$

3.11.6 Observe that

$$\mathrm{d}(x^2 + 2y^3) \wedge \mathrm{d}y = 2x\mathrm{d}x \wedge \mathrm{d}y.$$

Hence by the generalised Stokes' Theorem the integral equals

$$\int_{\{(x-2)^2+y^2\leq 4\}} 2x\mathrm{d}x \wedge \mathrm{d}y = \int_{-\pi/2}^{\pi/2} \int_0^{4\cos\theta} 2\rho^2 \cos\theta \mathrm{d}\rho \wedge \mathrm{d}\theta = 16\pi.$$

To do it directly, put $x - 2 = 2\cos t, y = 2\sin t, 0 \leq t \leq 2\pi$. Then the integral becomes

$$\begin{aligned} \int_0^{2\pi} ((2+2\cos t)^2 + 16\sin^3 t)\mathrm{d}2\sin t &= \int_0^{2\pi} (8\cos t + 16\cos^2 t \\ &\qquad +8\cos^3 t + 32\cos t\sin^3 t)\mathrm{d}t \\ &= 16\pi. \end{aligned}$$

3.11.7 At the intersection path

$$0 = x^2 + y^2 + z^2 - 2(x + y) = (2 - y)^2 + y^2 + z^2 - 4 = 2y^2 - 4y + z^2 = 2(y - 1)^2 + z^2 - 2,$$

which describes an ellipse on the yz-plane. Similarly we get $2(x - 1)^2 + z^2 = 2$ on the xz-plane. We have

$$\mathrm{d}\left(y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z\right) = \mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}z \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}z = -\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}z - \mathrm{d}z \wedge \mathrm{d}x.$$

Since $\mathrm{d}x \wedge \mathrm{d}y = 0$, by Stokes' Theorem the integral sought is

$$-\int_{2(y-1)^2+z^2\leq 2} \mathrm{d}y\mathrm{d}z - \int_{2(x-1)^2+z^2\leq 2} \mathrm{d}z\mathrm{d}x = -2\pi(\sqrt{2}).$$

(To evaluate the integrals you may resort to the fact that the area of the elliptical region $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} \leq 1$ is πab).

If we were to evaluate this integral directly, we would set

$$y = 1 + \cos\theta,\ z = \sqrt{2}\sin\theta, x = 2 - y = 1 - \cos\theta.$$

The integral becomes

$$\int_0^{2\pi} (1 + \cos\theta)\mathrm{d}(1 - \cos\theta) + \sqrt{2}\sin\theta\mathrm{d}(1 + \cos\theta) + (1 - \cos\theta)\mathrm{d}(\sqrt{2}\sin\theta)$$

which in turn

$$= \int_0^{2\pi} \sin\theta + \sin\theta\cos\theta - \sqrt{2} + \sqrt{2}\cos\theta\mathrm{d}\theta = -2\pi\sqrt{2}.$$

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